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A construction of spaces of compatible differential forms on cellular complexes

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Abstract

Given a cellular complex we construct spaces of differential forms which form a complex under the exterior derivative, which is isomorphic to the cochain complex of the cellular complex. The construction applies in particular to subsets of Euclidean space partitioned into polyhedra. The construction requires auxiliary spaces of differential forms on each cell, for which we provide two examples. When the cells are simplexes the construction can be used to recover the standard mixed finite element spaces. Furthermore the dual finite elements previously constructed by A. Buffa and the author on the barycentric refinement of a two-dimensional mesh, can be obtained in this way, and the method provides generalizations to all dimensions. In the framework of mimetic finite differences the construction provides a conforming reconstruction operator.

1 Introduction

Given a simplicial complex, such as a partition of some domain in $\mathbb{R}^3$ into tetrahedra with matching faces, there are well known finite-dimensional spaces of differential forms that are smooth on each simplex and compatible along interfaces and where moreover the space of k-forms has a natural basis indexed by the set of k-simplices. They are known as Whitney forms and as remarked in [4] they correspond to the lowest order mixed finite element spaces of [19][18]. The latter also provides constructions of such spaces that are canonical on partitions into some other shapes, such as cubes and cylinders with triangular base. Such finite element spaces are used for the simulation of physical phenomena ranging from fluid flow to electromagnetic waves and there is a well developed numerical analysis of the corresponding algorithms, see [6] [21] [17]. This analysis has been cast into the language of differential forms see e.g. [13], taking advantage of the machinery of homological algebra. In particular differential complexes and commuting diagrams are useful (see [1] for an exposition) and has lead to the construction of new spaces adapted to e.g. problems in elasticity [2].

The present paper is a continuation of this trend. For cell-decompositions that do not consist of canonical shapes such as the above there is, to the best of

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my knowledge, no known general construction of spaces of compatible differential forms forming a complex, where the k-forms have a basis indexed by the set of k-dimensional cells. For instance even if a mesh consists of simplices the dual mesh is not simplicial. The construction of spaces on the dual mesh was useful for some preconditioning purposes, motivating [8]. That construction was limited to two dimensions and given its ad-hoc nature it was not clear how it could be extended to higher dimensions. We notice furthermore that for polyhedral decompositions, a construction of a pair of spaces to treat diffusion problems was proposed in [7] but was not inserted into an appropriate complex of spaces. Other motivations for considering non-canonical polyhedra are given in that paper. In terms of mimetic finite differences, for which we refer to [3], we provide a compatible reconstruction operator valid for arbitrary cellular complexes. The construction requires some auxiliary spaces equipped with scalar products and given such data the method provides a unique solution to the above problem. For the auxiliary spaces one can use Whitney forms on a simplicial refinement of the cellular complex; then the method provides an answer to how so-called interior degrees of freedom can be eliminated. For the case of a cellular complex which is the dual of a simplicial one, the method reduces to the one of [8] in two dimensions and provides a generalization to higher dimensions. Even if the Whitney forms on the fine mesh provide superior approximation properties, the proposed method is useful in cases where spaces on the primal and dual mesh are required to match in some sense.

Questions regarding approximation properties of the proposed spaces depend on the choice of auxiliary ones as well as the scalar products, providing a topic for future research.

The paper is organized as follows: in §2 we provide some definitions, in §3 we provide the general construction and in §4 we provide some examples.

2 Definitions

We now define cellular complexes and provide some material on differential forms and cochains. The connection between the two is well studied see e.g. [20], but we give an exposition with notations that are consistent with those of [9][10].

Cellular complexes By a *cellular complex* we will mean a couple $(X, \mathcal{T})$, where X is a topological space and $\mathcal{T}$ is a finite set of finite subsets of X, equipped with a map $|\cdot| : \mathcal{T} \to \mathcal{P}(X)$ with the following properties:

- For each $T \in \mathcal{T}$, $|T|$ is a closed subset of X which can also be regarded as a smooth manifold with piecewise smooth boundary and which is homeomorphic to the closed unit ball of some Euclidean space. Moreover X is the union of the sets $|T|$ as T runs through $\mathcal{T}$ and a subset F of X is closed iff for each $T \in \mathcal{T}$, $F \cap |T|$ is closed in $|T|$ (when the latter is equipped with the manifold topology).

- We think of the sets $|T|$ as polyhedra with a set of distinguished vertices T. They are required to match in the following sense:

$$\forall T, T' \in \mathcal{T} \quad T \cap T' \in \mathcal{T}, \tag{1}$$

and:

$$\forall T, T' \in \mathcal{T} \quad |T \cap T'| = |T| \cap |T'|. \tag{2}$$

- The following condition on boundaries should also hold for each $T \in \mathcal{T}$:

$$\partial|T| = \cup\{|T'| \ : \ T' \in \mathcal{T},\ T' \subset T,\ T' \neq T\}. \tag{3}$$

Notations and vocabulary For each $T \in \mathcal{T}$ the dimension of the manifold $|T|$ is denoted $\dim T$ and we put for each $k \in \mathbb{N}$:

$$\mathcal{T}^k = \{T \in \mathcal{T} \ : \ \dim T = k\}. \tag{4}$$

With a slight abuse of notations, for any $T \in \mathcal{T}$ we put:

$$\partial T = \{T' \in \mathcal{T} \ : \ T' \subset T,\ T' \neq T\}. \tag{5}$$

A *refinement* of $(X, \mathcal{T})$ is a cellular complex $(X, \mathcal{T}')$ such that for each $T \in \mathcal{T}$:

$$|T|_{\mathcal{T}} = \cup\{|T'|_{\mathcal{T}'} \ : \ T' \in \mathcal{T}',\ |T'|_{\mathcal{T}'} \subset |T|_{\mathcal{T}}\}. \tag{6}$$

A *cellular subcomplex* of $(X, \mathcal{T})$ is a cellular complex $(X', \mathcal{T}')$ such that X' is a closed subset of X, $\mathcal{T}'$ is a subset of $\mathcal{T}$ and $|\cdot|_{\mathcal{T}'}$ is obtained by restricting $|\cdot|_{\mathcal{T}}$. In these circumstances we put :

$$|\mathcal{T}'| = \cup\{|T'| \ : \ T' \in \mathcal{T}'\}. \tag{7}$$

For instance, given a cellular complex $(X, \mathcal{T})$ and $T \in \mathcal{T}$, $(\partial|T|, \partial T)$ is a cellular subcomplex and eq. (3) can be written $\partial|T| = |\partial T|$. We also have the trivial cellular subcomplex $(|T|, \mathcal{P}(T) \cap \mathcal{T})$.

Fix now a cellular complex $(X, \mathcal{T})$. In the following we suppose that for each $T \in \mathcal{T}$, the manifold $|T|$ has been oriented. Given $T \in \mathcal{T}^{k+1}$ and $T' \in \mathcal{T}^k$, if $T' \subset T$ we define the *relative orientation* $\epsilon(T, T')$ to be 1 if T' is outward oriented compared with T and -1 if it is inward oriented. For all $T, T' \in \mathcal{T}$ not covered by this definition we put $\epsilon(T, T') = 0$.

Differential forms For each integer k we let Ω^k_c be the set of families $u = (u_T)_{T \in \mathcal{T}^k}$ such that for each $T \in \mathcal{T}$, u_T is a smooth k-form on $|T|$, subject to the following compatibility conditions: For each $T, T' \in \mathcal{T}$ such that $T' \subset T$, if we let $\rho : |T'| \to |T|$ be the canonical injection, we have $\rho^\star u_T = u_{T'}$, i. e. $u_{T'}$ is the pull-back of u_T. Thus u is determined by its values on the cells of maximal dimension and can be thought of as a globally defined piecewise smooth differential form satisfying compatibility conditions along cell interfaces. Consequently we will use the notation $u|_T = u_T$. For our applications it will be useful to denote by Ω^k the union of the spaces Ω^k_c associated with all refinements of $(X, \mathcal{T})$. In other words in Ω^k we allow also for piecewise smooth differential forms which are compatible with respect only to some refinement of the ("coarse") mesh $\mathcal{T}$.

Given $u \in \Omega^k_c$ the family $(\mathrm{d}u_T)_{T \in \mathcal{T}^k}$ is in Ω^{k+1}_c, as a consequence of the commutation of the exterior derivative with pullbacks, and will be denoted $\mathrm{d}u$. We thus obtain maps $\mathrm{d} : \Omega^k \to \Omega^{k+1}$. Since $\mathrm{d} \circ \mathrm{d} = 0$ as a map $\Omega^k \to \Omega^{k+2}$, the family $\Omega^\bullet = (\Omega^k)_{k \geq 0}$ is a complex. It is represented by a diagram:

$$0 \to \Omega^0 \to \Omega^1 \to \Omega^2 \to \cdots \tag{8}$$

Cochains For each k let C^k denote the set of maps $c : \mathcal{T}^k \to \mathbb{R}$. Such maps are called k-cochains. The coboundary operator $\delta : C^k \to C^{k+1}$ is defined by:

$$(\delta c)(T) = \sum_{T' \in \partial T \cap \mathcal{T}^k} \epsilon(T, T') c(T'). \tag{9}$$

One checks that $\delta \circ \delta = 0$ as a map $C^k \to C^{k+2}$, so that the family $C^\bullet = (C^k)_{k \geq 0}$ is a complex, called the cochain complex and represented by:

$$0 \to C^0 \to C^1 \to C^2 \to \cdots \tag{10}$$

The complex $C^\bullet$ can be related to $\Omega^\bullet$ as follows. For each $T \in \mathcal{T}^k$ let μ_T be the linear form on Ω^k defined by:

$$\mu_T : u \mapsto \int_T u|_T. \tag{11}$$

For each k we denote by $\mu^k : \Omega^k \to C^k$ the map defined by:

$$\mu^k : u \mapsto (\mu_T u)_{(T \in \mathcal{T}^k)}. \tag{12}$$

In the framework of mimetic finite differences [3], μ^k is called a reduction map, whereas in the setting of finite element computations, the linear forms μ_T are called degrees of freedom. As a consequence of Stokes theorem we have the well-known result:

Proposition 2.1 *For each k the following diagram commutes:*

$$\begin{array}{ccc} \Omega^k & \xrightarrow{\mathrm{d}} & \Omega^{k+1} \\ \mu^k \downarrow & & \mu^{k+1} \downarrow \\ C^k & \xrightarrow{\delta} & C^{k+1} \end{array} \tag{13}$$

It is easy to check that the maps $\mu^k : \Omega^k \to C^k$ are onto so that the above diagram uniquely determines $\delta : C^k \to C^{k+1}$.

3 Construction

The problem for which we propose a solution is to find a subcomplex $\Gamma^\bullet$ of $\Omega^\bullet$ – that is, spaces $\Gamma^k \subset \Omega^k$ such that $\mathrm{d}\Gamma^k \subset \Gamma^{k+1}$ – such that the restrictions $\mu^k : \Gamma^k \to C^k$ are isomorphisms. Such spaces would provide appropriate Galerkin spaces to solve variational problems involving the exterior derivative. In terms of finite elements we seek a subcomplex on which the degrees of freedom are unisolvent. In terms of mimetic finite differences, a right-inverse of μ_k such as the one provided by the subcomplex, is called a reconstruction operator.

For the construction of $\Gamma^\bullet$ we proceed by defining particular differential forms on a small cellular subcomplex and extending the domain of definition to larger cellular subcomplexes. Given a cellular subcomplex $(X', \mathcal{T}')$ of $(X, \mathcal{T})$ it will therefore be useful to denote by $\Omega^k(\mathcal{T}')$ the space of compatible piecewise smooth differential forms on X'.

We require for each k and each $T \in \mathcal{T}$, a subspace $A^k(T)$ of $\Omega^k(T)$ equipped with a scalar product, denoted $a(\cdot, \cdot)$. These spaces are subject to the following conditions.

- The exterior derivative should induce maps $\mathrm{d} : A^k(T) \to A^{k+1}(T)$, and given cells $T, T' \in \mathcal{T}$ such that $T' \subset T$, the canonical injection $\rho : |T'| \to |T|$ should by pullback (see e.g. [15]) induce maps $A^k(T) \to A^k(T')$.

- Given a cellular subcomplex $(X', \mathcal{T}')$ of $(X, \mathcal{T})$ we put:

$$A^k(\mathcal{T}') = \{u \in \Omega^k(\mathcal{T}') \ : \ \forall T \in \mathcal{T}' \quad u|_T \in A^k(T)\} \tag{14}$$

  We require that if $\rho : |\partial T| \to |T|$ denotes the canonical injection the pullback operator $\rho^\star : A^k(T) \to A^k(\partial T)$ is onto. In other words a compatible differential k-form u on $|\partial T|$ such that $u|_{T'}$ is in $A^k(T')$ for all $T' \in \partial T$, can be extended into T to an element of $A^k(T)$. We denote by $A^k(T, \partial T)$ the kernel of $\rho^\star : A^k(T) \to A^k(\partial T)$.

- Since $|T|$ has the topology of a ball we impose the following on cohomology groups (see e.g. [5]). The spaces $A^\bullet(T)$ form a complex under the exterior derivative:

$$0 \to A^0(T) \to \cdots \to A^k(T) \to A^{k+1}(T) \to \cdots \to A^{\dim T}(T) \to 0, \tag{15}$$

  for which the cohomology group $\mathrm{H}^k A^\bullet(T)$ should have dimension 1 for $k = 0$ and 0 otherwise. In particular $A^0(T)$ should contain the constant functions.

  By commutation of the exterior derivative with pullbacks, $A^\bullet(T, \partial T)$ is a subcomplex:

$$0 \to A^0(T, \partial T) \to \cdots \to A^k(T, \partial T) \to A^{k+1}(T, \partial T) \to \cdots \to A^{\dim T}(T, \partial T) \to 0, \tag{16}$$

  and we require that the cohomology group $\mathrm{H}^k A^\bullet(T, \partial T)$ has dimension 1 for $k = \dim T$ and 0 otherwise. Also $A^{\dim T}(T) = A^{\dim T}(T, \partial T)$ should contain a form with non-zero integral.

Lemma 3.1 *For each k, $T \in \mathcal{T}^k$ and $\alpha \in \mathbb{R}$ there is a unique element u of $A^{\dim T}(T)$ such that:*

$$\int_T u = \alpha \quad \textit{and} \quad \forall v \in A^{\dim T - 1}(T, \partial T) \quad a(u, \mathrm{d}v) = 0. \tag{17}$$

– *Proof:* The orthogonal of $\operatorname{im} \mathrm{d} : A^{\dim T-1}(T, \partial T) \to A^{\dim T}(T, \partial T)$ with respect to a is one-dimensional and contains an element with nonzero integral. □

With the above notations, the element with integral 1 will be denoted ω_T.

Lemma 3.2 *Pick a k. For each $T \in \mathcal{T}$ such that $\dim T > k$ if $u \in A^k(\partial T)$ there is a unique extension $u \in A^k(T)$ such that:*

$$\forall v \in A^k(T, \partial T) \quad a(\mathrm{d}u, \mathrm{d}v) = 0, \tag{18}$$

and:

$$\forall v \in A^{k-1}(T, \partial T) \quad a(u, \mathrm{d}v) = 0, \tag{19}$$

– *Proof:* Put $K = \operatorname{im} \mathrm{d} : A^{k-1}(T, \partial T) \to A^k(T, \partial T)$. Then on K, a is of course a scalar product but more importantly on its orthogonal $K^\perp$ in $A^k(T, \partial T)$ with respect to a, $a(\mathrm{d}\cdot, \mathrm{d}\cdot)$ is also a scalar product since $\dim T > k$ and therefore $K = \ker \mathrm{d} : A^k(T, \partial T) \to A^{k+1}(T, \partial T)$. Pick now $u_0 \in A^k(T)$, an arbitrary extension of u. If u_1 and u_2 are in $K^\perp$ and K respectively then $u = u_0 + u_1 + u_2$ solves our problem if and only if:

$$\forall v \in K^\perp \quad a(\mathrm{d}u_1, \mathrm{d}v) = -a(\mathrm{d}u_0, \mathrm{d}v), \tag{20}$$

and:

$$\forall v \in K \quad a(u_2, v) = -a(u_0, v). \tag{21}$$

This gives existence and uniqueness. □

The construction we propose is the following: Fix a k and a $T \in \mathcal{T}^k$. We will construct a $\lambda_T \in \Omega^k$ attached to T.

- First (using the first lemma above) put $\lambda_T|_T = \omega_T$ and for each $T' \in \mathcal{T}^k$ such that $T' \neq T$ we put $\lambda_T|_{T'} = 0$. Of course λ_T is set to zero also on cells of dimension $i < k$.

- Second (using the second lemma above), supposing λ_T has been defined on all cells of dimension up to some $i \geq k$ we define $\lambda_T|_{T'}$ on a cell $T' \in \mathcal{T}^{i+1}$ to be the unique element $u \in A^k(T')$ with $u|_{\partial T'}$ given by $\lambda_T|_{\partial T'}$ and such that:

$$\forall v \in A^k(T', \partial T') \quad a(\mathrm{d}u, \mathrm{d}v) = 0, \tag{22}$$

and:

$$\forall v \in A^{k-1}(T', \partial T') \quad a(u, \mathrm{d}v) = 0, \tag{23}$$

We now put:

$$\Gamma^k = \operatorname{span}\{\lambda_T \ : \ T \in \mathcal{T}^k\}. \tag{24}$$

Since for $T, T' \in \mathcal{T}^k$ we have $\mu_{T'} \lambda_T = \delta_{TT'}$ where the last symbol is the Kronecker delta, the family (λ_T) indexed by $T \in \mathcal{T}^k$ is a basis of Γ^k, and the induced map $\mu^k : \Gamma^k \to C^k$ is an isomorphism. Moreover:

Proposition 3.3 *The exterior derivative induces maps* $\mathrm{d} : \Gamma^k \to \Gamma^{k+1}$.

– *Proof:* We remark that for each k the elements of Γ^k are those $u \in \Omega^k$ whose restriction to any k-cell T is proportional to ω_T and which are extended inductively by the second lemma.

It is therefore enough to check that if u is an element of Γ^k then the restriction of $\mathrm{d}u$ to a $(k+1)$-dimensional cell T is proportional to ω_T. But $u|_T \in A^k(T)$ satisfies:

$$\forall v \in A^k(T, \partial T) \quad a(\mathrm{d}u, \mathrm{d}v) = 0. \tag{25}$$

Hence by the first lemma $\mathrm{d}u|_T = (\int_T \mathrm{d}u)\omega_T$. □

Proposition 3.4 *We have:*

$$\sum_{i \in \mathcal{T}^0} \lambda_i = 1. \tag{26}$$

– *Proof:* Define $u \in \Gamma^0$ by:

$$u = \sum_{i \in \mathcal{T}^0} \lambda_i. \tag{27}$$

We prove that $u = 1$ by induction on the dimension of the cells. For a 0-dimensional cell T it is of course true that $u|_T = 1$. Suppose now it has been proved that $u|_T = 1$ for all cells T of dimension $\leq k$ and consider a $(k+1)$-dimensional cell T. The constant function on T equal to 1 is an element of $A^0(T)$ whose boundary values are 1 and whose exterior derivative is 0. By the uniqueness proved in the second lemma above, we therefore have $u|_T = 1$. □

It is therefore tempting to believe that the family $(\lambda_i)_{i \in \mathcal{T}^0}$ is a partition of unity. However, in general, the functions λ_i might take negative values. Indeed as we shall see in the next section, one can use finite element functions on a refinement of $\mathcal{T}$ to construct the spaces $A^0(T)$ and use L^2 products for a. Then the constructed functions are so-called discrete harmonic on each cell, but it is known that the (refined) mesh needs to satisfy additional requirements for discrete maximum principles to hold [11].

4 Examples

Simplicial refinements Fix a cellular complex $(X, \mathcal{T})$ and consider a refinement $(X, \mathcal{T}')$ consisting of simplexes, so that $(X, \mathcal{T}')$ is actually a simplicial complex. If we denote by $C^\bullet(\mathcal{T}')$ its cochain complex there is a canonical construction of $\Gamma^\bullet(\mathcal{T}')$. The elements of $\Gamma^k(\mathcal{T}')$ are called Whitney forms, see e.g. [14] and we choose the basis (λ'_T), indexed by $T \in \mathcal{T}'^k$ satisfying, for all $T, T' \in \mathcal{T}'^k$:

$$\int_{T'} \lambda_T = \delta_{TT'}. \tag{28}$$

We then define for $T \in \mathcal{T}$, $A^k(T)$ to be the Whitney forms on $|T|$ with respect to the refinement $\mathcal{T}'$ restricted to T. For the scalar products $a(\cdot, \cdot)$ one can take the one which makes the canonical basis orthonormal. This choice satisfies all the above requirements and therefore provides a complex denoted $\Gamma^\bullet(\mathcal{T}, \mathcal{T}')$ which is actually a subcomplex of $\Gamma^\bullet(\mathcal{T}')$.

If we start with a simplicial complex $(X, \mathcal{S})$ and let $(X, \mathcal{S}')$ be its barycentric refinement, in the sense of [23], then there is a cellular complex $(X, \mathcal{T})$ which is called the dual of $(X, \mathcal{S})$, such that $(X, \mathcal{S}')$ is a refinement of $(X, \mathcal{T})$ (the top-dimensional cells of $\mathcal{T}$ are neighborhoods of the vertices of $\mathcal{S}$, and vice-versa). The complex $\Gamma^\bullet(\mathcal{T}, \mathcal{S}')$ obtained with the above construction is then isomorphic to the chain complex of $\mathcal{S}$ (the dual of its cochain complex). It turns out that this construction coincides with the complex constructed by A. Buffa and the author in [8]. Checking the involved orthogonality conditions is straightforward because of the choice of scalar products. Other choices of scalar products (such as L^2 products) might yield better approximation properties. Moreover due to the ad-hoc nature of our definition we were limited to manifolds of dimension 2 whereas the present construction is valid in all dimensions.

Continuous elements Fix again a cellular complex $(X, \mathcal{T})$ and suppose that a Riemannian metric g has been given on each cell and that they are compatible on interfaces between cells (in the sense of matching pullbacks, as for differential forms). Such Riemannian metrics were considered in [9] in the context of simplicial complexes. Such a Riemannian metric induces L^2 scalar products on differential forms, and on each cell we denote by $\mathrm{d}^\star$ the formal adjoint of d (see e.g. [24]). As a continuous analogue of the proposed construction we can let $A^k(T)$ consist of all smooth differential forms u such that u and $\mathrm{d}u$ have continuous extensions to the boundary, and such that the restrictions (pullbacks) to cells on the boundary have the same property. For $a(\cdot,\cdot)$ we take the L^2 products.

With the previous notations the continuous analogue of the proposed method is to denote for any k-cell T, ω_T the unique volume form on T such that:

$$\int_T \omega_T = 1 \quad \text{and} \quad \mathrm{d}^\star \omega_T = 0. \tag{29}$$

Moreover the extension of a form u from the boundary $|\partial T|$ to $|T|$ is characterized by the properties that on $|T|$:

$$\mathrm{d}^\star \mathrm{d}u = 0 \quad \text{and} \quad \mathrm{d}^\star u = 0. \tag{30}$$

We remark that since the maximum principle holds, $(\lambda_i)_{i \in \mathcal{T}^0}$ is a partition of unity in this case. However for general polyhedra the solutions to the above harmonic extension problems do not have explicit expressions which limits the usefulness of the corresponding spaces. However for many canonical shapes one has explicit solutions and it appears that one can recover many of the standard lowest order finite elements in this way. In particular this is the case in three-dimensional domains for a mesh consisting of tetrahedra: one recovers the lowest order Whitney forms or Nédélec's lowest order finite elements of the first family.

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