EIGENPROBLEM FOR JACOBI MATRICES: HYPERGEOMETRIC SERIES SOLUTION

VADIM B. KUZNETSOV AND EVGENY K. SKLYANIN

Abstract. We study the perturbative power-series expansions of the eigenvalues and eigenvectors of a general tridiagonal (Jacobi) matrix of dimension d. The (small) expansion parameters are being the entries of the two diagonals of length $d-1$ sandwiching the principal diagonal, which gives the unperturbed spectrum.

The solution is found explicitly in terms of multivariable (Horn-type) hypergeometric series of $3d-5$ variables in the generic case, or $2d-3$ variables for the eigenvalue growing from a corner matrix element. To derive the result, we first rewrite the spectral problem for a Jacobi matrix as an equivalent system of cubic equations, which are then resolved by the application of the multivariable Lagrange inversion formula.

Explicit formulae are also found for any monomial composed of eigenvector's components.

Contents

1. Introduction 1
2. Lagrange equations 3
3. Jacobian 8
4. Hypergeometric series 12
5. Corner eigenvalue 14
6. Discussion 15
References 15

1. Introduction

There are many applications of the Lagrange inversion formula for power series, from finding zeros of algebraic equations to deriving non-trivial identities for hypergeometric functions [1, 2].

The problem of finding series expansions for zeros of polynomials has a long history. In the classical paper [3], Birkeland developed the original approach through the Lagrange inversion formula. An advantage of this approach is that it equally gives not only the zeros but also their powers as explicit hypergeometric series.

Later on, an alternative approach was suggested by Mayr in [4], who derived the relevant hypergeometric series as solutions of some PDE's satisfied by the zeros as functions of the coefficients of the polynomial (small expansion variables).

Modern interpretation [5] is in terms of Gel'fand-Kapranov-Zelevinsky hypergeometric functions [6, 7, 8]. This approach can be applied to finding solutions of general systems of algebraic equations.

We adopt here the approach through the Lagrange inversion formula, as it gives us the quickest way to the desired result.

The main aim of this paper is to derive a complete solution of the spectral problem for an arbitrary finite $(d \times d)$ Jacobi matrix M. Specifically, for each eigenvalue and the corresponding eigenvector, it is a special system of d quadratic equations for d unknowns. We consider the off-diagonal matrix elements to be small, so that the whole problem looks as a perturbation of a diagonal matrix. We rewrite the spectral problem for a Jacobi matrix in an equivalent form as a *larger* system of special (Lagrange-form) $3d-5$ *cubic* equations, which are then inverted by the application of the multivariable Lagrange inversion formula. The solution is given explicitly in terms of multivariable (Horn-type) hypergeometric series of $3d-5$ variables. In the case of a corner eigenvalue the number of expansion variables drops to $2d-3$.

Consider a tridiagonal (Jacobi) matrix of order d

$$M=\begin{pmatrix} \alpha_1 & \beta_1 & & & & \\ \gamma_1 & \alpha_2 & \beta_2 & & & \\ & \cdots\cdots & \cdots\cdots & & & \\ & & \gamma_{k-1} & \alpha_k & \beta_k & \\ & & & \cdots\cdots & \cdots\cdots & \\ & & & & \gamma_{d-1} & \alpha_d \end{pmatrix} \tag{1.1}$$

and the corresponding eigenproblem $MV=\Lambda V$ for the eigenvector V and the eigenvalue Λ.

Assume that the off-diagonal elements β_k, γ_k are the small parameters of the power expansion and that α_k are distinct. In the zeroth approximation $\beta_k=\gamma_k=0$ the matrix M is diagonal, its eigenvalues and eigenvectors being α_k and $V^{(k)}$, respectively, where the components $V_j^{(k)}$ of $V^{(k)}$ can be chosen as $V_j^{(k)}=\delta_{jk}$. It is obvious then that for the small values of β_k, γ_k the eigenvalues Λ_k, $k=1,\dots,d$, are distinct and can be numbered in such a way that

$$\Lambda_k=\alpha_k+\text{higher order terms.} \tag{1.2}$$

We also choose to normalize the eigenvector $V^{(k)}$

$$MV^{(k)}=\Lambda_k V^{(k)} \tag{1.3}$$

by the condition

$$V_k^{(k)}=1 \tag{1.4}$$

for its k^{th} component. Therefore, the remaining components must vanish in the zeroth approximation

$$V_j^{(k)}=0+\text{higher order terms,} \qquad j\neq k. \tag{1.5}$$

The eigenvalue problem (1.3) together with the normalization condition (1.4) form a system of d algebraic (quadratic) equations for the eigenvalue Λ_k and the components $V_j^{(k)}$ of the eigenvector $V^{(k)}$ defining them as algebraic functions of the parameters α, β, γ. The conditions (1.2) and (1.5) fix uniquely the branches of the multivalued algebraic functions for small values of β and γ.

The problem we solve in the present paper is to find an effective way to construct explicit expressions for the coefficients of the power series expansions for the eigenvalues Λ_k and the components of the eigenvectors $V_j^{(k)}$. This problem falls into a large subject of expanding solutions of algebraic equations and of systems of algebraic equations into power series. It has been extensively treated in the literature. Despite a huge progress of the modern approach to this problem which generated a large body of knowledge about the generic case, we are not aware of any detailed analysis of the case of systems of equations arising from Jacobi matrix spectral problem. To deal with this special case of much physical interest, we follow the original idea of Birkeland [3] to use a variant of Lagrange inversion formula, rather then the later approach making use of differential equations.

We shall use the following variant of the multivariable Lagrange inversion theorem [9, 10]. Let boldface letters denote vectors $\boldsymbol{\xi} \equiv (\xi_1, \dots, \xi_D)$ and multi-indices $\boldsymbol{\xi}^{\boldsymbol{q}} \equiv \xi_1^{q_1} \dots \xi_D^{q_D}$. Let $[\boldsymbol{\xi}^{\boldsymbol{q}}]h(\boldsymbol{\xi})$ denote the coefficient at $\boldsymbol{\xi}^{\boldsymbol{q}}$ in the power series $h(\boldsymbol{\xi})$.

Theorem 1.1. *Let* $\boldsymbol{\xi} = (\xi_1, \dots, \xi_D)$*,* $\boldsymbol{\eta} = (\eta_1, \dots, \eta_D)$*. Let* $\chi(\boldsymbol{\eta})$ *and* $\varphi_i(\boldsymbol{\eta})$*,* $i = 1, \dots, D$*, be formal power series in* $\boldsymbol{\eta}$ *such that* $\varphi_i(\mathbf{0}) \neq 0\,\forall i$*. Then the system of* D *equations*

$$\xi_i = \frac{\eta_i}{\varphi_i(\boldsymbol{\eta})}, \qquad i = 1, \dots, D, \tag{1.6}$$

defines uniquely $\eta_i(\boldsymbol{\xi})$*,* $i = 1, \dots, D$*, as formal power series in* $\boldsymbol{\xi}$*. Moreover, the expansion for* $\chi\big(\boldsymbol{\eta}(\boldsymbol{\xi})\big)$ *is given by the formula*

$$[\boldsymbol{\xi}^{\boldsymbol{q}}]\chi\big(\boldsymbol{\eta}(\boldsymbol{\xi})\big) = [\boldsymbol{\eta}^{\boldsymbol{q}}]\chi(\boldsymbol{\eta})\boldsymbol{\varphi}^{\boldsymbol{q}}(\boldsymbol{\eta})J(\boldsymbol{\eta}), \tag{1.7}$$

where J *is the Jacobian*

$$J = \det\left(\mathbf{1} - \frac{\partial \log \boldsymbol{\varphi}}{\partial \log \boldsymbol{\eta}}\right) = \det\left|\delta_{jk} - \frac{\eta_k}{\varphi_j}\frac{\partial \varphi_j(\boldsymbol{\eta})}{\partial \eta_k}\right|. \tag{1.8}$$

When applying the Lagrange formula (1.7), the major complication comes from the Jacobian J, which may be difficult to compute. Fortunately, for our particular problem the Jacobian can be calculated explicitly.

The paper is organised as follows. In section 2 we write down the set of quadratic equations defining the eigenvalue Λ and the components V_j of the eigenvector V, transform them into the form which is convenient for studying, identify the combinations of the small parameters which serve as the expansion variables, and rewrite the equations again in the form which allows us to apply Lagrange's inversion formula. In section 3 we calculate an important ingredient of Lagrange's formula: the Jacobian J. In section 4 we put together all the ingredients of the Lagrange formula and produce explicit expressions for all Λ_k's and $V_j^{(k)}$'s as sums of power series which we recognize as Horn-type multivariable hypergeometric series. In section 5 we describe the simplification of our results for the special case $k = 1$ (or $k = d$) when the eigenvalue Λ stems from a corner of the matrix M. The last section contains a discussion of possible applications and extensions of our result.

2. Lagrange equations

From now on we shall fix the value of k and concentrate on studying the single eigenvalue $\Lambda_k \equiv \lambda$ and the corresponding eigenvector $V^{(k)} \equiv v$. We shall change our notation accordingly, to simplify the calculations. Set $r \equiv d - k$ and $\widetilde{r} \equiv k - 1$, so that $d = r + \widetilde{r} + 1$. Let $\Lambda_k = a_k + \lambda$, $V_j^{(k)} = \widetilde{v}_{k-j}$ and $\alpha_j = \widetilde{a}_{k-j}$ for $j = 1, \dots, k$, and $V_j^{(k)} = v_{j-k}$ and $\alpha_j = a_{j-k}$ for $j = k, \dots, d$, so that $V_k^{(k)} = v_0 = \widetilde{v}_0$ and $\alpha_k = a_0 = \widetilde{a}_0$. Respectively, let $\beta_j = \widetilde{b}_{k-j-1}$ and $\alpha_j = \widetilde{a}_{k-j-1}$ for $j = 1, \dots, k-1$, and $\beta_j = \beta_{j-k}$ and $\alpha_j = \alpha_{j-k}$ for $j = k, \dots, d-1$. Without loss of generality we can set $\alpha_k = a_0 = \widetilde{a}_0 = 0$. The normalisation condition (1.4) implies that $V_k^{(k)} = v_0 = \widetilde{v}_0 = 1$.

As a result, the eigenvalue problem (1.3) takes the following, more symmetric form:

$$\begin{pmatrix}
\widetilde{a}_{\widetilde{r}} & \widetilde{b}_{\widetilde{r}-1} & & & & & & & \\
 & \cdots & \cdots & & & & & & \\
 & \widetilde{c}_i & \widetilde{a}_i & \widetilde{b}_{i-1} & & & & & \\
 & & \cdots & \cdots & & & & & \\
 & & & \widetilde{c}_1 & \widetilde{a}_1 & \widetilde{b}_0 & & & \\
 & & & & \widetilde{c}_0 & 0 & b_0 & & \\
 & & & & & c_0 & a_1 & b_1 & \\
 & & & & & & \cdots & \cdots & \\
 & & & & & & c_{i-1} & a_i & b_i \\
 & & & & & & & \cdots & \cdots \\
 & & & & & & & c_{r-1} & a_r
\end{pmatrix}
\begin{pmatrix} \widetilde{v}_{\widetilde{r}} \\ \cdots \\ \widetilde{v}_i \\ \cdots \\ \widetilde{v}_1 \\ 1 \\ v_1 \\ \cdots \\ v_i \\ \cdots \\ v_r \end{pmatrix}
= \lambda
\begin{pmatrix} \widetilde{v}_{\widetilde{r}} \\ \cdots \\ \widetilde{v}_i \\ \cdots \\ \widetilde{v}_1 \\ 1 \\ v_1 \\ \cdots \\ v_i \\ \cdots \\ v_r \end{pmatrix}, \tag{2.1}$$

The numbers $\widetilde{r}, r = 0, 1, \ldots$, measure the distance of an eigenvalue from the corners of the matrix. The cases when $\widetilde{r}, r = 0, 1$ are slightly special, we shall remark about those in due course, all other cases with $\widetilde{r}, r \geq 2$ are generic.

Expressing λ from the 'central' ('zeroth') row as

$$\lambda = \widetilde{c}_0 \widetilde{v}_1 + b_0 v_1 \tag{2.2}$$

and substituting it into the remaining rows we get the set of $r + \widetilde{r}$ equations

$$c_{i-1} v_{i-1} + a_i v_i + b_i v_{i+1} = \widetilde{c}_0 \widetilde{v}_1 v_i + b_0 v_1 v_i, \qquad i = 1, \ldots, r, \tag{2.3a}$$

$$\widetilde{c}_i \widetilde{v}_{i+1} + \widetilde{a}_i \widetilde{v}_i + \widetilde{b}_{i-1} \widetilde{v}_{i-1} = \widetilde{c}_0 \widetilde{v}_1 \widetilde{v}_i + b_0 v_1 \widetilde{v}_i \qquad i = 1, \ldots, \widetilde{r}, \tag{2.3b}$$

where we assume that $v_0 \equiv \widetilde{v}_0 = 1$, $b_r = 0$, $\widetilde{c}_{\widetilde{r}} = 0$. The formula (2.2) eliminates the eigenvalue λ by expressing it in terms of the two eigenvalue's components. From now on, the variables v_i and $\widetilde{v}_i$ will be our only $r + \widetilde{r}$ unknowns satisfying $r + \widetilde{r}$ quadratic equations (2.3a)–(2.3b).

Note that the equations (2.3) are invariant with respect to the involution

$$r \leftrightarrow \widetilde{r}, \quad v_i \leftrightarrow \widetilde{v}_i, \quad a_i \leftrightarrow \widetilde{a}_i, \quad b_i \leftrightarrow \widetilde{c}_i, \quad c_i \leftrightarrow \widetilde{b}_i \tag{2.4}$$

which we refer to as the *tilde-symmetry.* This is seen in (2.1) as the symmetry with respect to reflection about the 'anti-diagonal' passing through the 'central' zero. Of course, r and $\widetilde{r}$ interchange, therefore the reflected matrix will be centred differently, if $r \neq \widetilde{r}$.

The next step is to rescale the variables v_i and $\widetilde{v}_i$ in order to identify the convenient combinations of the expansion parameters. Setting $b_0 = \ldots = b_{r-1} = 0$ and $\widetilde{c}_0 = \ldots = \widetilde{c}_{\widetilde{r}-1} = 0$ and denoting the corresponding values of v_i and $\widetilde{v}_i$ as, respectively, v_i^0 and $\widetilde{v}_i^0$ we get the equations

$$c_{i-1} v_{i-1}^0 + a_i v_i^0 = 0 \qquad \Longrightarrow \qquad v_i^0 = -\frac{c_{i-1}}{a_i}\, v_{i-1}^0, \tag{2.5a}$$

$$\widetilde{a}_i \widetilde{v}_i^0 + \widetilde{b}_{i-1} \widetilde{v}_{i-1}^0 = 0 \qquad \Longrightarrow \qquad \widetilde{v}_i = -\frac{\widetilde{b}_{i-1}}{\widetilde{a}_i}\, \widetilde{v}_{i-1}^0, \tag{2.5b}$$

which can be solved recursively yielding

$$v_i^0 = (-1)^i \, \frac{c_0 \ldots c_{i-1}}{a_1 \ldots a_i}, \qquad i = 1, \ldots, r, \tag{2.6a}$$

$$\widetilde{v}_i^0 = (-1)^i \, \frac{\widetilde{b}_0 \ldots \widetilde{b}_{i-1}}{\widetilde{a}_1 \ldots \widetilde{a}_i}, \qquad i = 1, \ldots, \widetilde{r}. \tag{2.6b}$$

Let us rescale v_i and $\widetilde{v}_i$ by the formulae

$$v_i = v_i^0 u_i = (-1)^i \frac{c_0 \dots c_{i-1}}{a_1 \dots a_i} u_i, \qquad i = 1, \dots, r, \tag{2.7a}$$

$$\widetilde{v}_i = \widetilde{v}_i^0 \widetilde{u}_i = (-1)^i \frac{\widetilde{b}_0 \dots \widetilde{b}_{i-1}}{\widetilde{a}_1 \dots \widetilde{a}_i} \widetilde{u}_i, \qquad i = 1, \dots, \widetilde{r} \tag{2.7b}$$

and rewrite the equations (2.3) in terms of u_i, $\widetilde{u}_i$:

$$u_i = u_{i-1} - \frac{b_0 c_0}{a_1 a_i} u_1 u_i - \frac{\widetilde{b}_0 \widetilde{c}_0}{\widetilde{a}_1 a_i} \widetilde{u}_1 u_i + \frac{b_i c_i}{a_i a_{i+1}} u_{i+1}, \qquad i = 1, \dots, r, \tag{2.8a}$$

$$\widetilde{u}_i = \widetilde{u}_{i-1} - \frac{\widetilde{b}_0 \widetilde{c}_0}{\widetilde{a}_1 \widetilde{a}_i} \widetilde{u}_1 \widetilde{u}_i - \frac{b_0 c_0}{a_1 \widetilde{a}_1} u_1 \widetilde{u}_i + \frac{\widetilde{b}_i \widetilde{c}_i}{\widetilde{a}_i \widetilde{a}_{i+1}} \widetilde{u}_{i+1}, \qquad i = 1, \dots, \widetilde{r}, \tag{2.8b}$$

where we assume $u_0 = \widetilde{u}_0 = 1$, $b_r = \widetilde{c}_r = 0$.

The equations (2.8) contain the parameters $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \widetilde{\boldsymbol{a}}, \widetilde{\boldsymbol{b}}, \widetilde{\boldsymbol{c}}$ in the specific combinations which are convenient to use as the expansion parameters. Note that b_i and c_i enter only through the product $b_i c_i$ (same for $\widetilde{b}_i \widetilde{c}_i$). Introduce the variables x_i, y_i, z_i

$$x_i = \frac{b_0 c_0}{a_1 a_{i+1}}, \qquad i = 0, \dots, r-1, \tag{2.9a}$$

$$y_i = \frac{\widetilde{b}_0 \widetilde{c}_0}{\widetilde{a}_1 a_{i+1}}, \qquad i = 0, \dots, r-1, \tag{2.9b}$$

$$z_i = \frac{b_{i+1} c_{i+1}}{a_{i+1} a_{i+2}}, \qquad i = 0, \dots, r-2, \tag{2.9c}$$

(altogether $r + r + (r-1) = 3r - 1$ variables) as well as their tilde-analogs

$$\widetilde{x}_i = \frac{\widetilde{b}_0 \widetilde{c}_0}{\widetilde{a}_1 \widetilde{a}_{i+1}}, \qquad i = 0, \dots, \widetilde{r}-1, \tag{2.10a}$$

$$\widetilde{y}_i = \frac{b_0 c_0}{a_1 \widetilde{a}_{i+1}}, \qquad i = 0, \dots, \widetilde{r}-1, \tag{2.10b}$$

$$\widetilde{z}_i = \frac{\widetilde{b}_{i+1} \widetilde{c}_{i+1}}{\widetilde{a}_{i+1} \widetilde{a}_{i+2}}, \qquad i = 0, \dots, \widetilde{r}-2, \tag{2.10c}$$

(altogether $\widetilde{r} + \widetilde{r} + (\widetilde{r} - 1) = 3\widetilde{r} - 1$ variables). The total number of variables is then $D = 3r + 3\widetilde{r} - 2 = 3d - 5$. Let us remark that the variables $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}, \widetilde{\boldsymbol{x}}, \widetilde{\boldsymbol{y}}, \widetilde{\boldsymbol{z}})$ are not independent, as there can be only $2r + 2\widetilde{r}$ independent variables because $\boldsymbol{b}, \boldsymbol{c}$ $(\widetilde{\boldsymbol{b}}, \widetilde{\boldsymbol{c}})$ enter only in the products $b_i c_i$ $(\widetilde{b}_i \widetilde{c}_i)$. Nevertheless, for our purposes we shall use them as independent variables. In what follows we shall attach to them the same number, $D = 3d - 5$, dependent variables, thereby reformulating the original system (2.3) of $d - 1$ equations as the one embedded into a larger system. One reason for introducing this larger system is that it can consequently be put into the form suitable for application of the Lagrange inversion theorem. There are, in fact, many more reasons to believe that the above choice of the independent variables is not only a natural one but is also the right one. A strong argument comes from the fact that with such a choice of the variables the $(3d-5)$-dimensional Jacobian can be found explicitly, which in its turn results in explicit representations of all unknowns in terms of multiple hypergeometric series with explicit coefficients. The final result therefore shall finally demonstrate that the above choice of variables is relevant for the solution of the problem. Exactly these variables enter as arguments of the associated hypergeometric series.

The equations (2.8) are simplified now to the form

$$u_i = u_{i-1} - x_{i-1}u_1u_i - y_{i-1}\widetilde{u}_1u_i + z_{i-1}u_{i+1}, \qquad i = 1,\dots,r, \tag{2.11a}$$

$$\widetilde{u}_i = \widetilde{u}_{i-1} - \widetilde{x}_{i-1}\widetilde{u}_1\widetilde{u}_i - \widetilde{y}_{i-1}u_1\widetilde{u}_i + \widetilde{z}_{i-1}\widetilde{u}_{i+1}, \qquad i = 1,\dots,\widetilde{r}, \tag{2.11b}$$

where we assume $u_0 = \widetilde{u}_0 = 1$, $z_{r-1} = \widetilde{z}_{\widetilde{r}-1} = 0$. Since $b_i, \widetilde{b}_i, c_i, \widetilde{c}_i$ are small parameters, so are $x_i, y_i, z_i, \widetilde{x}_i, \widetilde{y}_i, \widetilde{z}_i$. The iteration of the equations (2.11) with the initial values $u_i^0 = \widetilde{u}_i^0 = 1$ produces formal power series expansions of u_i and $\widetilde{u}_i$. The tilde-symmetry (2.4) for the equations (2.11) takes the form

$$r \leftrightarrow \widetilde{r}, \quad u_i \leftrightarrow \widetilde{u}_i, \quad x_i \leftrightarrow \widetilde{x}_i, \quad y_i \leftrightarrow \widetilde{y}_i, \quad z_i \leftrightarrow \widetilde{z}_i \tag{2.12}$$

In order to find explicitly the coefficients of the power series for u_i and $\widetilde{u}_i$ we shall use the Lagrange inversion formula. In the notation of the Theorem 1.1, the expansion variables vector $\boldsymbol{\xi}$ is composed of the 6 sets of variables $\boldsymbol{\xi} = (\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}, \widetilde{\boldsymbol{x}}, \widetilde{\boldsymbol{y}}, \widetilde{\boldsymbol{z}})$. To meet the theorem's premises we have to introduce also the vector $\boldsymbol{\eta}$ composed, respectively, of the 6 matching sets of expandable quantities $\boldsymbol{\eta} = (\boldsymbol{s}, \boldsymbol{t}, \boldsymbol{w}, \widetilde{\boldsymbol{s}}, \widetilde{\boldsymbol{t}}, \widetilde{\boldsymbol{w}})$:

$$s_i \sim x_i \qquad i = 0,\dots,r-1; \qquad \widetilde{s}_j \sim \widetilde{x}_j \qquad j = 0,\dots,\widetilde{r}-1, \tag{2.13a}$$

$$t_i \sim y_i \qquad i = 0,\dots,r-1; \qquad \widetilde{t}_j \sim \widetilde{y}_j \qquad j = 0,\dots,\widetilde{r}-1, \tag{2.13b}$$

$$w_i \sim z_i \qquad i = 0,\dots,r-2; \qquad \widetilde{w}_i \sim \widetilde{z}_i \qquad i = 0,\dots,\widetilde{r}-2. \tag{2.13c}$$

Note that the variables $\boldsymbol{u}\widetilde{\boldsymbol{u}}$ cannot be used as $\boldsymbol{\eta}$ because they have nonzero limits for $\boldsymbol{\xi} \to \boldsymbol{0}$, therefore they have to be completed by small factors. Besides, the number of the variables $\boldsymbol{u}\widetilde{\boldsymbol{u}}$ is only $r + \widetilde{r} = d - 1$, so we need $2d - 4$ extra variables to match the number $D = 3d - 5$ of expansion variables. Actually, it appears that the arithmetic is completely different and the right choice of variables corresponds to the following splitting of the number $D \equiv 3d - 5 = (2d - 2) + (d - 3)$, with the first set of $2d - 2$ variables for a larger system coming from only two variables, u_1 and $\widetilde{u}_1$. Define s_i, t_i, $\widetilde{s}_i$, $\widetilde{t}_i$ as

$$s_i = x_i u_1, \qquad t_i = y_i \widetilde{u}_1, \qquad i = 0,\dots,r-1, \tag{2.14a}$$

$$\widetilde{s}_i = \widetilde{x}_i \widetilde{u}_1, \qquad \widetilde{t}_i = \widetilde{y}_i u_1, \qquad i = 0,\dots,\widetilde{r}-1. \tag{2.14b}$$

The second set of $d - 3$ extra variables $\boldsymbol{w}, \widetilde{\boldsymbol{w}}$ must be defined in terms of the (NB: two-step) ratios of the variables u_i and $\widetilde{u}_i$, respectively:

$$w_i = z_i \frac{u_{i+2}}{u_i}, \qquad i = 0,\dots,r-2, \tag{2.15a}$$

$$\widetilde{w}_i = \widetilde{z}_i \frac{\widetilde{u}_{i+2}}{\widetilde{u}_i}, \qquad i = 0,\dots,\widetilde{r}-2. \tag{2.15b}$$

From $u_0 = \widetilde{u}_0 = 1$ it follows that

$$w_0 = z_0 u_2, \tag{2.16a}$$

$$\widetilde{w}_0 = \widetilde{z}_0 \widetilde{u}_2. \tag{2.16b}$$

Solving the equations (2.15)–(2.16) we obtain the expressions for $u_2,\dots,u_r$ in terms of w_j and for $\widetilde{u}_2,\dots,\widetilde{u}_{\widetilde{r}}$ in terms of $\widetilde{w}_j$:

$$u_{2j} = \frac{w_{2j-2}\dots w_0}{z_{2j-2}\dots z_0}, \qquad u_{2j+1} = \frac{w_{2j-1}\dots w_1 s_0}{z_{2j-1}\dots z_1 x_0}, \tag{2.17a}$$

$$\widetilde{u}_{2j} = \frac{\widetilde{w}_{2j-2}\dots w_0}{\widetilde{z}_{2j-2}\dots \widetilde{z}_0}, \qquad \widetilde{u}_{2j+1} = \frac{\widetilde{w}_{2j-1}\dots \widetilde{w}_1 \widetilde{s}_0}{\widetilde{z}_{2j-1}\dots \widetilde{z}_1 \widetilde{x}_0}. \tag{2.17b}$$

Proposition 2.1. *The set of equations (2.11), (2.14) and (2.15) is equivalent to the set of Lagrange-type (1.6) equations*

$$x_i = \frac{s_i}{f_i}, \qquad y_i = \frac{t_i}{g_i}, \qquad i = 0, \dots, r-1, \qquad z_j = \frac{w_j}{h_j}, \qquad j = 0, \dots, r-2, \tag{2.18a}$$

$$\widetilde{x}_i = \frac{\widetilde{s}_i}{\widetilde{f}_i}, \qquad \widetilde{y}_i = \frac{\widetilde{t}_i}{\widetilde{g}_i}, \qquad i = 0, \dots, \widetilde{r}-1, \qquad \widetilde{z}_j = \frac{\widetilde{w}_j}{\widetilde{h}_j}, \qquad j = 0, \dots, \widetilde{r}-2, \tag{2.18b}$$

where

$$f_0 = \ldots = f_{r-1} = \widetilde{g}_0 = \ldots = \widetilde{g}_{\widetilde{r}-1} = \frac{1+w_0}{1+s_0+t_0}, \tag{2.19a}$$

$$h_i = \frac{(1+w_i)(1+w_{i+1})}{(1+s_i+t_i)(1+s_{i+1}+t_{i+1})}, \qquad i = 0, \dots, r-2 \qquad (w_{r-1} \equiv 0), \tag{2.19b}$$

$$\widetilde{f}_0 = \ldots = \widetilde{f}_{\widetilde{r}-1} = g_0 = \ldots = g_{r-1} = \frac{1+\widetilde{w}_0}{1+\widetilde{s}_0+\widetilde{t}_0}, \tag{2.20a}$$

$$\widetilde{h}_i = \frac{(1+\widetilde{w}_i)(1+\widetilde{w}_{i+1})}{(1+\widetilde{s}_i+\widetilde{t}_i)(1+\widetilde{s}_{i+1}+\widetilde{t}_{i+1})}, \qquad i = 0, \dots, \widetilde{r}-2 \qquad (\widetilde{w}_{\widetilde{r}-1} \equiv 0). \tag{2.20b}$$

In the notation of the Theorem 1.1, we have

$$\boldsymbol{\xi} = (\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}, \widetilde{\boldsymbol{x}}, \widetilde{\boldsymbol{y}}, \widetilde{\boldsymbol{z}}), \quad \boldsymbol{\eta} = (\boldsymbol{s}, \boldsymbol{t}, \boldsymbol{w}, \widetilde{\boldsymbol{s}}, \widetilde{\boldsymbol{t}}, \widetilde{\boldsymbol{w}}), \quad \boldsymbol{\varphi} = (\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}, \widetilde{\boldsymbol{f}}, \widetilde{\boldsymbol{g}}, \widetilde{\boldsymbol{h}}). \tag{2.21}$$

Proof. Consider the equation (2.11a) for u_{i+1}:

$$u_{i+1} = u_i - x_i u_1 u_{i+1} - y_i \widetilde{u}_1 u_{i+1} + z_i u_{i+2}. \tag{2.22}$$

Note that $z_i u_{i+2} = w_i u_i$ by virtue of (2.15a). Having rearranged the terms we get

$$u_{i+1}(1 + x_i u_1 + y_i \widetilde{u}_1) = u_i(1 + w_i). \tag{2.23}$$

From (2.14a) it follows that $x_i u_1 = s_i$ and $y_i \widetilde{u}_1 = t_i$. Therefore,

$$\frac{u_i}{u_{i+1}} = \frac{1+s_i+t_i}{1+w_i}. \tag{2.24}$$

Multiplying the equalities (2.24) for the index i and $i+1$ we get

$$\frac{u_i}{u_{i+2}} = \frac{(1+s_i+t_i)(1+s_{i+1}+t_{i+1})}{(1+w_i)(1+w_{i+1})}. \tag{2.25}$$

It remains to replace u_i/u_{i+2} with z_i/w_i from the equality (2.15a), and we obtain the equality of the form $z_i = w_i/h_i$, see (2.18a), where

$$h_i = \frac{(1+w_i)(1+w_{i+1})}{(1+s_i+t_i)(1+s_{i+1}+t_{i+1})}, \qquad i = 0, \dots, r-2, \tag{2.26}$$

and it is assumed that $w_{r-1} = 0$.

The special case of the equation (2.11a) for $i = 1$,

$$u_1 = 1 - x_0 u_1^2 - y_0 \widetilde{u}_1 u_1 + z_0 u_2, \tag{2.27}$$

should be treated separately. Similarly to the general case, we replace $z_0 u_2$ with w_0 using (2.16a), and substitute $u_1 = s_0/x_0$ from (2.14a) and $\widetilde{u}_1 = \widetilde{s}_0/\widetilde{x}_0$ from (2.14b). Then we replace $y_0 \widetilde{s}_0$ with $\widetilde{x}_0 t_0$, using both (2.14a) and (2.14b). As a result, the equation takes the form

$$s_0(1 + s_0 + t_0) = x_0(1 + w_0) \tag{2.28}$$

or, equivalently, the form $x_0 = s_0/f_0$, see (2.18a) for $i = 0$, where

$$f_0 = \frac{1 + w_0}{1 + s_0 + t_0}. \tag{2.29}$$

From (2.14a) and (2.14b) it also follows that

$$f_i = \frac{s_i}{x_i} = u_1 = \frac{\widetilde{t}_i}{\widetilde{y}_i} = \widetilde{g}_i = \frac{s_0}{x_0} = f_0. \tag{2.30}$$

The remaining half of the equations is obtained by the tilde-symmetry (2.12). ■

As was said before, the case of a corner eigenvalue, i.e. when $r = 0$ or $\widetilde{r} = 0$, and the next one, i.e. when $r = 1$ or $\widetilde{r} = 1$, are slightly special. Let us remark that when one applies the formulae from the above theorem describing the generic case, i.e. when both $\widetilde{r}, r \geq 2$, to such cases one has to remember that

- for a corner eigenvalue, say $r = 0$, there are no untilded variables and there are also *no* variables $\widetilde{\boldsymbol{y}}, \widetilde{\boldsymbol{t}}, \widetilde{\boldsymbol{g}}$, so that in this case we only have $2d - 3 = 2(\widetilde{r} + 1) - 3 = 2\widetilde{r} - 1$ variables, therefore one must disregard (2.19) entirely and put $\widetilde{t}_j \equiv 0$, $j = 0, \ldots, \widetilde{r} - 1$, in (2.20);
- for the immediate next eigenvalue, say $r = 1$, all variables are present but $\boldsymbol{w}$, so that one must set $w_0 \equiv 0$ in (2.19a) and disregard (2.19b) entirely, there are $3d - 5 = 3(\widetilde{r} + 2) - 5 = 3\widetilde{r} + 1$ variables in this case.

All these modifications are easily seen from the expressions (2.9)–(2.10) of the small parameters $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}, \widetilde{\boldsymbol{x}}, \widetilde{\boldsymbol{y}}, \widetilde{\boldsymbol{z}})$ in terms of the initial small parameters $(\boldsymbol{b}, \boldsymbol{c}, \widetilde{\boldsymbol{b}}, \widetilde{\boldsymbol{c}})$.

3. JACOBIAN

In this section only, we use the ordering of variables which is different from the one used above: $\boldsymbol{\xi} = (\widetilde{\boldsymbol{x}}, \widetilde{\boldsymbol{y}}, \boldsymbol{x}, \boldsymbol{y}, \widetilde{\boldsymbol{z}}, \boldsymbol{z})$, $\boldsymbol{\eta} = (\widetilde{\boldsymbol{s}}, \widetilde{\boldsymbol{t}}, \boldsymbol{s}, \boldsymbol{t}, \widetilde{\boldsymbol{w}}, \boldsymbol{w})$, $\boldsymbol{\varphi} = (\widetilde{\boldsymbol{f}}, \widetilde{\boldsymbol{g}}, \boldsymbol{f}, \boldsymbol{g}, \widetilde{\boldsymbol{h}}, \boldsymbol{h})$.

Theorem 3.1. *The Jacobian J defined by (1.8) can be expressed as*

$$J = S\widetilde{S} - T\widetilde{T}, \tag{3.1}$$

where

$$S = \frac{1}{W}\left(1 + \sum_{j=0}^{r-1} \frac{s_j(1 + w_j)}{1 + s_j + t_j} \prod_{k=0}^{j-1} w_k\right), \qquad T = \frac{1}{W} \sum_{j=0}^{r-1} \frac{t_j(1 + w_j)}{1 + s_j + t_j} \prod_{k=0}^{j-1} w_k, \tag{3.2}$$

and

$$W = \prod_{k=0}^{r-2} (1 + w_k). \tag{3.3}$$

Here we adopt the following agreement: whenever a sum has the upper limit smaller than the lower one then its value should be taken as 0. Also, any product in a similar case should be taken as 1. The tildes in the above formulae refer to replacing s_j, t_j, w_j and r by their tilded versions. We must also assume that, as always, $\widetilde{w}_{\widetilde{r}-1} \equiv 0$ and $w_{r-1} \equiv 0$ (similar to the stated in the Proposition 2.1).

Let us remark that the number of terms in J (3.1) is $r\widetilde{r} + (r + 1)(\widetilde{r} + 1) = 2r\widetilde{r} + r + \widetilde{r} + 1 = 2dk - 2k^2 + 2k - d$ (recall that $d = \widetilde{r} + r + 1$ and $k = \widetilde{r} + 1$).

Proof. Consider the rows of the matrix $\mathcal{J} = \delta_{jk} - \frac{\eta_j}{\varphi_k} \frac{\partial \varphi_k}{\partial \eta_j}$:

$$\left(1 + \tfrac{\widetilde{s}_0}{1+\widetilde{s}_0+\widetilde{t}_0}, \tfrac{\widetilde{s}_0}{1+\widetilde{s}_0+\widetilde{t}_0}, \ldots, 0, \ldots, 0, \ldots, \tfrac{\widetilde{s}_0}{1+\widetilde{s}_0+\widetilde{t}_0}, \ldots, \tfrac{\widetilde{s}_0}{1+\widetilde{s}_0+\widetilde{t}_0}, 0, \ldots, 0, \ldots\right) \tag{3.4}$$

$$\left(0, 1, 0, \ldots, 0, \ldots, 0, \ldots, 0, \ldots, \tfrac{\widetilde{s}_1}{1+\widetilde{s}_1+\widetilde{t}_1}, \tfrac{\widetilde{s}_1}{1+\widetilde{s}_1+\widetilde{t}_1}, 0, \ldots, 0, \ldots\right)$$

$$\left(0, 0, 1, 0, \ldots, 0, \ldots, 0, \ldots, 0, \ldots, 0, \tfrac{\widetilde{s}_2}{1+\widetilde{s}_2+\widetilde{t}_2}, \tfrac{\widetilde{s}_2}{1+\widetilde{s}_2+\widetilde{t}_2}, 0, \ldots, 0, \ldots\right)$$

$$\vdots$$

$$\begin{gathered}
\left(\tfrac{\widetilde{t}_0}{1+\widetilde{s}_0+\widetilde{t}_0},\dots,1,0,\dots,0,\dots,\tfrac{\widetilde{t}_0}{1+\widetilde{s}_0+\widetilde{t}_0},\dots,\tfrac{\widetilde{t}_0}{1+\widetilde{s}_0+\widetilde{t}_0},0,\dots,0,\dots\right) \\
\left(0,\dots,0,1,0,\dots,0,\dots,0,\dots,\tfrac{\widetilde{t}_1}{1+\widetilde{s}_1+\widetilde{t}_1},\tfrac{\widetilde{t}_1}{1+\widetilde{s}_1+\widetilde{t}_1},0,\dots,0,\dots\right) \\
\left(0,\dots,0,0,1,0,\dots,0,\dots,0,\dots,0,\tfrac{\widetilde{t}_2}{1+\widetilde{s}_2+\widetilde{t}_2},\tfrac{\widetilde{t}_2}{1+\widetilde{s}_2+\widetilde{t}_2},0,\dots,0,\dots\right) \\
\vdots
\end{gathered} \tag{3.5}$$

$$\begin{gathered}
\left(0,\dots,\tfrac{s_0}{1+s_0+t_0},\dots,1+\tfrac{s_0}{1+s_0+t_0},\tfrac{s_0}{1+s_0+t_0},\dots,0,\dots,0,\dots,\tfrac{s_0}{1+s_0+t_0},0,\dots\right) \\
\left(0,\dots,0,\dots,0,1,0,\dots,0,\dots,0,\dots,\tfrac{s_1}{1+s_1+t_1},\tfrac{s_1}{1+s_1+t_1},0,\dots\right) \\
\left(0,\dots,0,\dots,0,0,1,0,\dots,0,\dots,0,\dots,0,\tfrac{s_2}{1+s_2+t_2},\tfrac{s_2}{1+s_2+t_2},0,\dots\right) \\
\vdots
\end{gathered} \tag{3.6}$$

$$\begin{gathered}
\left(0,\dots,\tfrac{t_0}{1+s_0+t_0},\dots,\tfrac{t_0}{1+s_0+t_0},\dots,1,0,\dots,0,\dots,\tfrac{t_0}{1+s_0+t_0},0,\dots\right) \\
\left(0,\dots,0,\dots,0,\dots,0,1,0,\dots,0,\dots,\tfrac{t_1}{1+s_1+t_1},\tfrac{t_1}{1+s_1+t_1},0,\dots\right) \\
\left(0,\dots,0,\dots,0,\dots,0,0,1,0,\dots,0,\dots,0,\tfrac{t_2}{1+s_2+t_2},\tfrac{t_2}{1+s_2+t_2},0,\dots\right) \\
\vdots
\end{gathered} \tag{3.7}$$

$$\begin{gathered}
\left(-\tfrac{\widetilde{w}_0}{1+\widetilde{w}_0},\dots,0,\dots,0,\dots,-\tfrac{\widetilde{w}_0}{1+\widetilde{w}_0},\dots,\tfrac{1}{1+\widetilde{w}_0},0,\dots,0,\dots\right) \\
\left(0,\dots,0,\dots,0,\dots,0,\dots,-\tfrac{\widetilde{w}_1}{1+\widetilde{w}_1},\tfrac{1}{1+\widetilde{w}_1},0,\dots,0,\dots\right) \\
\left(0,\dots,0,\dots,0,\dots,0,\dots,0,-\tfrac{\widetilde{w}_2}{1+\widetilde{w}_2},\tfrac{1}{1+\widetilde{w}_2},0,\dots,0,\dots\right) \\
\vdots
\end{gathered} \tag{3.8}$$

$$\begin{gathered}
\left(0,\dots,-\tfrac{w_0}{1+w_0},\dots,-\tfrac{w_0}{1+w_0},\dots,0,\dots,0,\dots,\tfrac{1}{1+w_0},0,\dots\right) \\
\left(0,\dots,0,\dots,0,\dots,0,\dots,0,\dots,-\tfrac{w_1}{1+w_1},\tfrac{1}{1+w_1},0,\dots\right) \\
\left(0,\dots,0,\dots,0,\dots,0,\dots,0,\dots,0,-\tfrac{w_2}{1+w_2},\tfrac{1}{1+w_2},0,\dots\right) \\
\vdots
\end{gathered} \tag{3.9}$$

First lines of each of the four first sub-sets above, which correspond to the variables $\widetilde{s}_0$ (3.4), $\widetilde{t}_0$ (3.5), s_0 (3.6) and t_0 (3.7), can be replaced by simpler versions (without changing the determinant) by adding a multiple of the row associated with $\widetilde{w}_0$ (3.8) or w_0 (3.9). They become as follows:

$$\begin{gathered}
\left(1,0,\dots,0,\dots,0,\dots,0,\dots,\tfrac{\widetilde{s}_0(1+\widetilde{w}_0)}{(1+\widetilde{s}_0+\widetilde{t}_0)\widetilde{w}_0},0,\dots,0,\dots\right) \\
\left(0,\dots,1,0,\dots,0,\dots,0,\dots,\tfrac{\widetilde{t}_0(1+\widetilde{w}_0)}{(1+\widetilde{s}_0+\widetilde{t}_0)\widetilde{w}_0},0,\dots,0,\dots\right) \\
\left(0,\dots,0,\dots,1,0,\dots,0,\dots,0,\dots,\tfrac{s_0(1+w_0)}{(1+s_0+t_0)w_0},0,\dots\right) \\
\left(0,\dots,0,\dots,0,\dots,1,0,\dots,0,\dots,\tfrac{t_0(1+w_0)}{(1+s_0+t_0)w_0},0,\dots\right)
\end{gathered}$$

After the above replacement, the $(3\widetilde{r}+3r-2)\times(3\widetilde{r}+3r-2)$ matrix $\mathcal{J}=\delta_{jk}-\frac{\eta_j}{\varphi_k}\frac{\partial\varphi_k}{\partial\eta_j}$ acquires the following block-matrix form:

$$\mathcal{J}=\begin{pmatrix}\mathbf{1}_{(2\widetilde{r}+2r)\times(2\widetilde{r}+2r)} & B\\ C & D\end{pmatrix},\qquad \text{whence}\qquad J\equiv\det\mathcal{J}=\det(D-CB),$$

where the matrix B is equal to

$$\begin{pmatrix}
\frac{\widetilde{s}_0(1+\widetilde{w}_0)}{(1+\widetilde{s}_0+\widetilde{t}_0)\widetilde{w}_0}, & 0, & 0, & 0,\ldots, & 0, & 0\\
\frac{\widetilde{s}_1}{1+\widetilde{s}_1+\widetilde{t}_1}, & \frac{\widetilde{s}_1}{1+\widetilde{s}_1+\widetilde{t}_1}, & 0, & 0,\ldots, & 0, & 0\\
0, & \frac{\widetilde{s}_2}{1+\widetilde{s}_2+\widetilde{t}_2}, & \frac{\widetilde{s}_2}{1+\widetilde{s}_2+\widetilde{t}_2}, & 0,\ldots, & 0, & 0\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\
0,\ldots, & 0, & \frac{\widetilde{s}_{\widetilde{r}-1}}{1+\widetilde{s}_{\widetilde{r}-1}+\widetilde{t}_{\widetilde{r}-1}}, & 0,\ldots, & 0, & 0\\
\frac{\widetilde{t}_0(1+\widetilde{w}_0)}{(1+\widetilde{s}_0+\widetilde{t}_0)\widetilde{w}_0}, & 0, & 0, & 0,\ldots, & 0, & 0\\
\frac{\widetilde{t}_1}{1+\widetilde{s}_1+\widetilde{t}_1}, & \frac{\widetilde{t}_1}{1+\widetilde{s}_1+\widetilde{t}_1}, & 0, & 0,\ldots, & 0, & 0\\
0, & \frac{\widetilde{t}_2}{1+\widetilde{s}_2+\widetilde{t}_2}, & \frac{\widetilde{t}_2}{1+\widetilde{s}_2+\widetilde{t}_2}, & 0,\ldots, & 0, & 0\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\
0,\ldots, & 0, & \frac{\widetilde{t}_{\widetilde{r}-1}}{1+\widetilde{s}_{\widetilde{r}-1}+\widetilde{t}_{\widetilde{r}-1}}, & 0,\ldots, & 0, & 0\\
0,\ldots, & 0, & 0, & \frac{s_0(1+w_0)}{(1+s_0+t_0)w_0}, & 0, & 0\\
0,\ldots, & 0, & 0, & \frac{s_1}{1+s_1+t_1}, & \frac{s_1}{1+s_1+t_1}, & 0\\
0,\ldots, & 0, & 0, & 0, & \frac{s_2}{1+s_2+t_2}, & \frac{s_2}{1+s_2+t_2}\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\
0,\ldots, & 0, & 0, & 0,\ldots, & 0, & \frac{s_{r-1}}{1+s_{r-1}+t_{r-1}}\\
0,\ldots, & 0, & 0, & \frac{t_0(1+w_0)}{(1+s_0+t_0)w_0}, & 0, & 0\\
0,\ldots, & 0, & 0, & \frac{t_1}{1+s_1+t_1}, & \frac{t_1}{1+s_1+t_1}, & 0\\
0,\ldots, & 0, & 0, & 0, & \frac{t_2}{1+s_2+t_2}, & \frac{t_2}{1+s_2+t_2}\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\
0,\ldots, & 0, & 0, & 0,\ldots, & 0, & \frac{t_{r-1}}{1+s_{r-1}+t_{r-1}}
\end{pmatrix},$$

$$C=\begin{pmatrix}
-\frac{\widetilde{w}_0}{1+\widetilde{w}_0},\ldots, & 0,\ldots, & 0,\ldots, & -\frac{\widetilde{w}_0}{1+\widetilde{w}_0},\ldots\\
0,\ldots, & 0,\ldots, & 0,\ldots, & 0,\ldots\\
\vdots & \vdots & \vdots & \vdots\\
0,\ldots, & 0,\ldots, & 0,\ldots, & 0,\ldots\\
0,\ldots, & -\frac{w_0}{1+w_0},\ldots, & -\frac{w_0}{1+w_0},\ldots, & 0,\ldots\\
0,\ldots, & 0,\ldots, & 0,\ldots, & 0,\ldots\\
\vdots & \vdots & \vdots & \vdots\\
0,\ldots, & 0,\ldots, & 0,\ldots, & 0,\ldots
\end{pmatrix},$$

$$
D=\begin{pmatrix}
\frac{1}{1+\widetilde{w}_0}, & 0, & 0,\dots, & 0, & 0,\dots, & 0\\
-\frac{\widetilde{w}_1}{1+\widetilde{w}_1}, & \frac{1}{1+\widetilde{w}_1}, & 0,\dots, & 0, & 0,\dots, & 0\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\
0,\dots, & -\frac{\widetilde{w}_{\widetilde{r}-2}}{1+\widetilde{w}_{\widetilde{r}-2}}, & \frac{1}{1+\widetilde{w}_{\widetilde{r}-2}}, & 0, & 0,\dots, & 0\\
0,\dots, & 0, & 0, & \frac{1}{1+w_0}, & 0, & 0\\
0,\dots, & 0, & 0, & -\frac{w_1}{1+w_1}, & \frac{1}{1+w_1}, & 0\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\
0,\dots, & 0, & 0, & 0,\dots, & -\frac{w_{r-2}}{1+w_{r-2}}, & \frac{1}{1+w_{r-2}}
\end{pmatrix}. \tag{3.10}
$$

Therefore, the calculation of the Jacobian has been reduced to finding an explicit formula for the determinant of a smaller, $(\widetilde{r}+r-2)\times(\widetilde{r}+r-2)$, matrix $D-CB$. Subtraction of the product CB does only change two rows of the triangular matrix D, those associated with the variables $\widetilde{w}_0$ and w_0. They now become

$$
\begin{aligned}
&\tfrac{\widetilde{w}_0}{1+\widetilde{w}_0}\Big(\tfrac{1}{\widetilde{w}_0}+\tfrac{\widetilde{s}_0(1+\widetilde{w}_0)}{(1+\widetilde{s}_0+\widetilde{t}_0)\widetilde{w}_0}+\tfrac{\widetilde{s}_1}{1+\widetilde{s}_1+\widetilde{t}_1},\\
&\qquad\tfrac{\widetilde{s}_1}{1+\widetilde{s}_1+\widetilde{t}_1}+\tfrac{\widetilde{s}_2}{1+\widetilde{s}_2+\widetilde{t}_2},\dots,\tfrac{\widetilde{s}_{\widetilde{r}-2}}{1+\widetilde{s}_{\widetilde{r}-2}+\widetilde{t}_{\widetilde{r}-2}}+\tfrac{\widetilde{s}_{\widetilde{r}-1}}{1+\widetilde{s}_{\widetilde{r}-1}+\widetilde{t}_{\widetilde{r}-1}},\\
&\quad\tfrac{t_0(1+w_0)}{(1+s_0+t_0)w_0}+\tfrac{t_1}{1+s_1+t_1},\\
&\qquad\qquad\tfrac{t_1}{1+s_1+t_1}+\tfrac{t_2}{1+s_2+t_2},\dots,\tfrac{t_{r-2}}{1+s_{r-2}+t_{r-2}}+\tfrac{t_{r-1}}{1+s_{r-1}+t_{r-1}}\Big)
\end{aligned}
$$

and

$$
\begin{aligned}
&\tfrac{w_0}{1+w_0}\Big(\tfrac{\widetilde{t}_0(1+\widetilde{w}_0)}{(1+\widetilde{s}_0+\widetilde{t}_0)\widetilde{w}_0}+\tfrac{\widetilde{t}_1}{1+\widetilde{s}_1+\widetilde{t}_1},\\
&\qquad\tfrac{\widetilde{t}_1}{1+\widetilde{s}_1+\widetilde{t}_1}+\tfrac{\widetilde{t}_2}{1+\widetilde{s}_2+\widetilde{t}_2},\dots,\tfrac{\widetilde{t}_{\widetilde{r}-2}}{1+\widetilde{s}_{\widetilde{r}-2}+\widetilde{t}_{\widetilde{r}-2}}+\tfrac{\widetilde{t}_{\widetilde{r}-1}}{1+\widetilde{s}_{\widetilde{r}-1}+\widetilde{t}_{\widetilde{r}-1}},\\
&\quad\tfrac{1}{w_0}+\tfrac{s_0(1+w_0)}{(1+s_0+t_0)w_0}+\tfrac{s_1}{1+s_1+t_1},\\
&\qquad\qquad\tfrac{s_1}{1+s_1+t_1}+\tfrac{s_2}{1+s_2+t_2},\dots,\tfrac{s_{r-2}}{1+s_{r-2}+t_{r-2}}+\tfrac{s_{r-1}}{1+s_{r-1}+t_{r-1}}\Big),
\end{aligned}
$$

respectively. The rest of the rows in the matrix $D-CB$ are the same as in the matrix D above, that is they all have only two non-zero elements: one is on the diagonal and the other one is to the left of it. By adding multiples of the columns, without changing the determinant, we can arrange that all those non-diagonal entries in the selected $(\widetilde{r}-2)+(r-2)$ rows vanish, allowing us to compute the determinant by reducing it down to a 2×2 determinant.

Indeed, multiply the last column of the matrix $D-CB$ (look at the formula (3.10)) by w_{r-2} and add the result to the penultimate column. The last row has now got only one (diagonal) element. Hence, we keep the factor $\frac{1}{1+w_{r-2}}$ and reduce the determinant to its minor, removing the last column and the last row. By repeating this process, we shall end up with the expression

$$
J=\frac{\widetilde{w}_0 w_0(\mathcal{S}\widetilde{\mathcal{S}}-\mathcal{T}\widetilde{\mathcal{T}})}{(1+\widetilde{w}_0)\cdots(1+\widetilde{w}_{\widetilde{r}-2})(1+w_0)\cdots(1+w_{r-2})},
$$

where

$$\begin{aligned}
\widetilde{\mathcal{S}} &= \tfrac{1}{\widetilde{w}_0} + \tfrac{\widetilde{s}_0(1+\widetilde{w}_0)}{(1+\widetilde{s}_0+\widetilde{t}_0)\widetilde{w}_0} + \tfrac{\widetilde{s}_1(1+\widetilde{w}_1)}{1+\widetilde{s}_1+\widetilde{t}_1} + \sum_{j=2}^{\widetilde{r}-2} \tfrac{\widetilde{s}_j(1+\widetilde{w}_j)\prod_{k=1}^{j-1}\widetilde{w}_k}{1+\widetilde{s}_j+\widetilde{t}_j} + \tfrac{\widetilde{s}_{\widetilde{r}-1}\prod_{k=1}^{\widetilde{r}-2}\widetilde{w}_k}{1+\widetilde{s}_{\widetilde{r}-1}+\widetilde{t}_{\widetilde{r}-1}},\\
\mathcal{T} &= \tfrac{t_0(1+w_0)}{(1+s_0+t_0)w_0} + \tfrac{t_1(1+w_1)}{1+s_1+t_1} + \sum_{j=2}^{r-2} \tfrac{t_j(1+w_j)\prod_{k=1}^{j-1}w_k}{1+s_j+t_j} + \tfrac{t_{r-1}\prod_{k=1}^{r-2}w_k}{1+s_{r-1}+t_{r-1}},\\
\widetilde{\mathcal{T}} &= \tfrac{\widetilde{t}_0(1+\widetilde{w}_0)}{(1+\widetilde{s}_0+\widetilde{t}_0)\widetilde{w}_0} + \tfrac{\widetilde{t}_1(1+\widetilde{w}_1)}{1+\widetilde{s}_1+\widetilde{t}_1} + \sum_{j=2}^{\widetilde{r}-2} \tfrac{\widetilde{t}_j(1+\widetilde{w}_j)\prod_{k=1}^{j-1}\widetilde{w}_k}{1+\widetilde{s}_j+\widetilde{t}_j} + \tfrac{\widetilde{t}_{\widetilde{r}-1}\prod_{k=1}^{\widetilde{r}-2}\widetilde{w}_k}{1+\widetilde{s}_{\widetilde{r}-1}+\widetilde{t}_{\widetilde{r}-1}},\\
\mathcal{S} &= \tfrac{1}{w_0} + \tfrac{s_0(1+w_0)}{(1+s_0+t_0)w_0} + \frac{s_1(1+w_1)}{1+s_1+t_1} + \sum_{j=2}^{r-2} \tfrac{s_j(1+w_j)\prod_{k=1}^{j-1}w_k}{1+s_j+t_j} + \tfrac{s_{r-1}\prod_{k=1}^{r-2}w_k}{1+s_{r-1}+t_{r-1}}.
\end{aligned}$$

This is apparently equivalent to the statement of the Theorem. ■

4. HYPERGEOMETRIC SERIES

Now we have all the ingredients for the right-hand side of the Lagrange formula (1.7) and can calculate the expansion coefficients. In this section we return to the initial ordering $\boldsymbol{\xi} = (\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}, \widetilde{\boldsymbol{x}}, \widetilde{\boldsymbol{y}}, \widetilde{\boldsymbol{z}})$, $\boldsymbol{\eta} = (\boldsymbol{s}, \boldsymbol{t}, \boldsymbol{w}, \widetilde{\boldsymbol{s}}, \widetilde{\boldsymbol{t}}, \widetilde{\boldsymbol{w}})$ and $\boldsymbol{\varphi} = (\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}, \widetilde{\boldsymbol{f}}, \widetilde{\boldsymbol{g}}, \widetilde{\boldsymbol{h}})$.

Let us choose the arbitrary function $\chi(\boldsymbol{\eta})$ in (1.7) to be the monomial $\chi(\boldsymbol{\eta}) = \boldsymbol{\eta}^{\boldsymbol{q}'} \equiv \boldsymbol{s}^{\boldsymbol{m}'}\boldsymbol{t}^{\boldsymbol{n}'}\boldsymbol{w}^{\boldsymbol{p}'}\widetilde{\boldsymbol{s}}^{\widetilde{\boldsymbol{m}}'}\widetilde{\boldsymbol{t}}^{\widetilde{\boldsymbol{n}}'}\widetilde{\boldsymbol{w}}^{\widetilde{\boldsymbol{p}}'}$.

Note that, due to the formulae (2.14) and (2.17), any monomial in $\boldsymbol{u}\widetilde{\boldsymbol{u}}$ can be expressed as a monomial in $\boldsymbol{\xi}$. The components v_i of the eigenvector v are, in turn, proportional to $\boldsymbol{u}\widetilde{\boldsymbol{u}}$ by (2.7). Thus, from the expansion of the monomial $\boldsymbol{\eta}^{\boldsymbol{q}'}$ in $\boldsymbol{\xi}$ one can derive the expansion of any monomial in v_j in $\boldsymbol{\xi}$ and, by (2.9) and (2.10), in $\boldsymbol{b}\boldsymbol{c}\widetilde{\boldsymbol{b}}\widetilde{\boldsymbol{c}}$. To get the expansions involving the eigenvalue λ one has to use (2.2).

Define the step function σ_{ij} as

$$\sigma_{ij} \equiv \begin{cases} 0, & i \le j \\ 1, & i > j \end{cases}. \tag{4.1}$$

We shall use the binomial and trinomial coefficients defined for integer m, n as

$$\binom{a}{n} \equiv [x^n](1+x)^a, \qquad \binom{a}{m,n} \equiv [x^m y^n](1+x+y)^a. \tag{4.2}$$

Note that the multinomial coefficients are evaluated as

$$\binom{a}{n} = \frac{a(a-1)\dots(a-n+1)}{n!}, \qquad \binom{a}{m,n} = \frac{a(a-1)\dots(a-m-n+1)}{m!n!} \tag{4.3}$$

for $m \ge 0$ and $n \ge 0$ and vanish if $m < 0$ or $n < 0$.

Let $|m| = m_0 + \ldots + m_{r-1}$ etc., and

$$p_{-1} \equiv |m| + |\widetilde{n}|, \qquad \widetilde{p}_{-1} \equiv |\widetilde{m}| + |n|. \tag{4.4}$$

Theorem 4.1. *The expansion coefficient of the monomial* $\chi = \boldsymbol{\eta}^{\boldsymbol{q}'} \equiv \boldsymbol{s}^{\boldsymbol{m}'}\boldsymbol{t}^{\boldsymbol{n}'}\boldsymbol{w}^{\boldsymbol{p}'}$ $\widetilde{\boldsymbol{s}}^{\widetilde{\boldsymbol{m}}'}\widetilde{\boldsymbol{t}}^{\widetilde{\boldsymbol{n}}'}\widetilde{\boldsymbol{w}}^{\widetilde{\boldsymbol{p}}'}$ *at the monomial* $\boldsymbol{\xi}^{\boldsymbol{q}} = \boldsymbol{x}^{\boldsymbol{m}}\boldsymbol{y}^{\boldsymbol{n}}\boldsymbol{z}^{\boldsymbol{p}}\widetilde{\boldsymbol{x}}^{\widetilde{\boldsymbol{m}}}\widetilde{\boldsymbol{y}}^{\widetilde{\boldsymbol{n}}}\widetilde{\boldsymbol{z}}^{\widetilde{\boldsymbol{p}}}$ *is given by*

$$\begin{aligned}
[\boldsymbol{\xi}^{\boldsymbol{q}}]\boldsymbol{\eta}^{\boldsymbol{q}'} &\equiv [\boldsymbol{x}^{\boldsymbol{m}}\boldsymbol{y}^{\boldsymbol{n}}\boldsymbol{z}^{\boldsymbol{p}}\widetilde{\boldsymbol{x}}^{\widetilde{\boldsymbol{m}}}\widetilde{\boldsymbol{y}}^{\widetilde{\boldsymbol{n}}}\widetilde{\boldsymbol{z}}^{\widetilde{\boldsymbol{p}}}]\boldsymbol{s}^{\boldsymbol{m}'}\boldsymbol{t}^{\boldsymbol{n}'}\boldsymbol{w}^{\boldsymbol{p}'}\widetilde{\boldsymbol{s}}^{\widetilde{\boldsymbol{m}}'}\widetilde{\boldsymbol{t}}^{\widetilde{\boldsymbol{n}}'}\widetilde{\boldsymbol{w}}^{\widetilde{\boldsymbol{p}}'}\\
&= \sum_{i=-1}^{r-1}\sum_{\widetilde{\imath}=-1}^{\widetilde{r}-1} \mathcal{A}_i^{\boldsymbol{mnp}}\widetilde{\mathcal{A}}_{\widetilde{\imath}}^{\widetilde{\boldsymbol{m}}\widetilde{\boldsymbol{n}}\widetilde{\boldsymbol{p}}} - \sum_{i=0}^{r-1}\sum_{\widetilde{\imath}=0}^{\widetilde{r}-1} \mathcal{B}_i^{\boldsymbol{mnp}}\widetilde{\mathcal{B}}_{\widetilde{\imath}}^{\widetilde{\boldsymbol{m}}\widetilde{\boldsymbol{n}}\widetilde{\boldsymbol{p}}},
\end{aligned} \tag{4.5}$$

where

$$\mathcal{A}_i^{\boldsymbol{mnp}} = \prod_{j=0}^{r-1} \binom{-p_{j-1} - p_j - \delta_{ij}}{m_j - m_j' - \delta_{ij},\, n_j - n_j'} \prod_{j=0}^{r-2} \binom{p_{j-1} + p_j + \delta_{ij} - 1}{p_j - p_j' - \sigma_{ij}}, \tag{4.6}$$

$$\mathcal{B}_i^{\boldsymbol{mnp}} = \prod_{j=0}^{r-1} \binom{-p_{j-1} - p_j - \delta_{ij}}{m_j - m'_j,\ n_j - n'_j - \delta_{ij}} \prod_{j=0}^{r-2} \binom{p_{j-1} + p_j + \delta_{ij} - 1}{p_j - p'_j - \sigma_{ij}} \tag{4.7}$$

(and respective tilded versions).

Proof. Using the definitions (2.19) and (2.20) we get for $\boldsymbol{q = (m, n, p, \widetilde{m}, \widetilde{n}, \widetilde{p})}$:

$$\begin{aligned}
\boldsymbol{\varphi^q(\eta)} &\equiv \boldsymbol{f^m g^n h^p \widetilde{f}^{\widetilde{m}} \widetilde{g}^{\widetilde{n}} \widetilde{h}^{\widetilde{p}}} \\
&= f_0^{m_0} \dots f_{r-1}^{m_{r-1}} g_0^{n_0} \dots g_{r-1}^{n_{r-1}} h_0^{p_0} \dots h_{r-2}^{p_{r-2}} \\
&\quad \times \widetilde{f}_0^{\widetilde{m}_0} \dots \widetilde{f}_{\widetilde{r}-1}^{\widetilde{m}_{\widetilde{r}-1}} \widetilde{g}_0^{\widetilde{n}_0} \dots \widetilde{g}_{\widetilde{r}-1}^{\widetilde{n}_{\widetilde{r}-1}} \widetilde{h}_0^{\widetilde{p}_0} \dots \widetilde{h}_{\widetilde{r}-2}^{\widetilde{p}_{\widetilde{r}-2}} \\
&= \left(\frac{1 + w_0}{1 + s_0 + t_0} \right)^{|m| + |\widetilde{n}| + p_0} \left(\prod_{j=1}^{r-1} \left(\frac{1 + w_j}{1 + s_j + t_j} \right)^{p_{j-1} + p_j} \right) \\
&\quad \times \left(\frac{1 + \widetilde{w}_0}{1 + \widetilde{s}_0 + \widetilde{t}_0} \right)^{|\widetilde{m}| + |n| + \widetilde{p}_0} \left(\prod_{j=1}^{\widetilde{r}-1} \left(\frac{1 + \widetilde{w}_j}{1 + \widetilde{s}_j + \widetilde{t}_j} \right)^{\widetilde{p}_{j-1} + \widetilde{p}_j} \right),
\end{aligned}$$

where we assume $w_{r-1} = \widetilde{w}_{\widetilde{r}-1} = p_{r-1} = \widetilde{p}_{\widetilde{r}-1} = 0$.

Using the shorthand notation (4.4) we can write down $\boldsymbol{\varphi^q(\eta)}$ in the compact form

$$\boldsymbol{\varphi^q(\eta)} = \left(\prod_{j=0}^{r-1} \left(\frac{1 + w_j}{1 + s_j + t_j} \right)^{p_{j-1} + p_j} \right) \left(\prod_{j=0}^{\widetilde{r}-1} \left(\frac{1 + \widetilde{w}_j}{1 + \widetilde{s}_j + \widetilde{t}_j} \right)^{\widetilde{p}_{j-1} + \widetilde{p}_j} \right).$$

Using the step function σ_{ij} we rewrite the expressions (3.2) for the ingredients of the Jacobian J in the following equivalent form:

$$T = \sum_{i=0}^{r-1} \prod_{j=0}^{r-1} \frac{w_j^{\sigma_{ij}} t_j^{\delta_{ij}}}{(1 + w_j)^{1 - \delta_{ij}} (1 + s_j + t_j)^{\delta_{ij}}},$$

$$S = \sum_{i=-1}^{r-1} \prod_{j=0}^{r-1} \frac{w_j^{\sigma_{ij}} s_j^{\delta_{ij}}}{(1 + w_j)^{1 - \delta_{ij}} (1 + s_j + t_j)^{\delta_{ij}}}.$$

Substituting the above expressions for χ, $\boldsymbol{\varphi^q}$ and J into the right-hand side of the Lagrange formula (1.7) we get

$$\chi \boldsymbol{\varphi^q} J = \sum_{i=-1}^{r-1} \sum_{\widetilde{\imath}=-1}^{\widetilde{r}-1} \mathcal{A}_i \widetilde{\mathcal{A}}_{\widetilde{\imath}} - \sum_{i=0}^{r-1} \sum_{\widetilde{\imath}=0}^{\widetilde{r}-1} \mathcal{B}_i \widetilde{\mathcal{B}}_{\widetilde{\imath}}, \tag{4.8}$$

where

$$\begin{aligned}
\mathcal{A}_i &= \boldsymbol{s^{m'} t^{n'} w^{p'}} \\
&\quad \times \left(\prod_{j=0}^{r-1} \left(\frac{1 + w_j}{1 + s_j + t_j} \right)^{p_{j-1} + p_j} \right) \\
&\quad \times \left(\prod_{j=0}^{r-1} \left(\frac{s_j}{1 + s_j + t_j} \right)^{\delta_{ij}} \frac{w_j^{\sigma_{ij}}}{(1 + w_j)^{1 - \delta_{ij}}} \right) \\
&= \prod_{j=0}^{r-1} s_j^{m'_j + \delta_{ij}} t_j^{n'_j} w_j^{p'_j + \sigma_{ij}} \\
&\quad \times (1 + s_j + t_j)^{-p_{j-1} - p_j - \delta_{ij}} (1 + w_j)^{p_{j-1} + p_j + \delta_{ij} - 1},
\end{aligned} \tag{4.9}$$

$$\begin{aligned}\mathcal{B}_i &= \prod_{j=0}^{r-1} s_j^{m'_j} t_j^{n'_j+\delta_{ij}} w_j^{p'_j+\sigma_{ij}} \\ &\quad \times (1+s_j+t_j)^{-p_{j-1}-p_j-\delta_{ij}} (1+w_j)^{p_{j-1}+p_j+\delta_{ij}-1} \qquad (4.10)\end{aligned}$$

and $\widetilde{\mathcal{A}}_{\tilde{\imath}}$, $\widetilde{\mathcal{B}}_{\tilde{\imath}}$ are their tilded versions.

It remains to take the coefficient of the resulting expression for $\chi\boldsymbol{\varphi}^{\boldsymbol{q}}J$ at the monomial $\boldsymbol{\eta}^{\boldsymbol{q}}$. Using the binomial and trinomial expansions (4.2) we get the expressions (4.6) and (4.7) for the expansion coefficients $\mathcal{A}_i^{\boldsymbol{mnp}} \equiv [\boldsymbol{s}^{\boldsymbol{m}}\boldsymbol{t}^{\boldsymbol{n}}\boldsymbol{w}^{\boldsymbol{p}}]\mathcal{A}_i$ and $\mathcal{B}_i^{\boldsymbol{mnp}} \equiv [\boldsymbol{s}^{\boldsymbol{m}}\boldsymbol{t}^{\boldsymbol{n}}\boldsymbol{w}^{\boldsymbol{p}}]\mathcal{B}_i$. The final expression (4.5) for $[\boldsymbol{\xi}^{\boldsymbol{q}}]\boldsymbol{\eta}^{\boldsymbol{q}'}$ follows then immediately. ■

We see from this final result that the problem is being solved in terms of Horn-type hypergeometric series, which are defined as arbitrary power series such that ratios of their shifted coefficients are rational functions of the indices.

In this paper we study only formal power series expansions. The analytic version of the Lagrange theorem [9] guarantees that our power series converge in an open neighbourhood around $\boldsymbol{\xi} = \boldsymbol{0}$. The precise description of the convergence domain of a multivariate power series is, however, a difficult task, see for example [8, 11], and we leave it to further study.

5. Corner eigenvalue

Let us consider a special case of the corner eigenvalue $\widetilde{r} = 0$. The problem corresponding to the other corner eigenvalue $r = 0$ can be treated by the tilde-symmetry. We have

$$\begin{pmatrix} 0 & b_0 & & & & & \\ c_0 & a_1 & b_1 & & & & \\ & & \cdots\cdots\cdots & & & & \\ & & & c_{i-1} & a_i & b_i & \\ & & & & \cdots\cdots\cdots & & \\ & & & & & c_{r-1} & a_r \end{pmatrix} \begin{pmatrix} 1 \\ v_1 \\ \cdots \\ v_i \\ \cdots \\ v_r \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ v_1 \\ \cdots \\ v_i \\ \cdots \\ v_r \end{pmatrix}. \tag{5.1}$$

The tilded variables are absent. Besides, the sequences y_i, resp. t_i are also absent. The total set of variables $\boldsymbol{\xi} = (\boldsymbol{x}, \boldsymbol{z})$ has the cardinality $(r)+(r-1) = 2r-1 = 2d-3$, and all the expansion variables are independent, in contrast with the generic case. Respectively, $\boldsymbol{\eta} = (\boldsymbol{s}, \boldsymbol{w})$ and $\boldsymbol{\varphi} = (\boldsymbol{f}, \boldsymbol{h})$, where

$$f_0 = \ldots = f_{r-1} = \frac{1+w_0}{1+s_0}, \qquad h_i = \frac{(1+w_i)(1+w_{i+1})}{(1+s_i)(1+s_{i+1})}, \qquad i = 0, \ldots, r-2. \tag{5.2}$$

The expression (3.1) for the jacobian J simplifies to

$$J = S = \frac{1}{\prod_{k=0}^{r-2}(1+w_k)} \left(1 + \sum_{j=0}^{r-1} \frac{s_j(1+w_j)}{1+s_j} \prod_{k=0}^{j-1} w_k\right) \tag{5.3}$$

and contains thus only $r+1 = d$ terms. The analog of the formula (4.8) is

$$\chi\boldsymbol{\varphi}^{\boldsymbol{q}}J = \sum_{i=-1}^{r-1} \mathcal{A}_i, \tag{5.4}$$

where

$$\mathcal{A}_i = \prod_{j=0}^{r-1} s_j^{m'_j+\delta_{ij}} w_j^{p'_j+\sigma_{ij}} (1+s_j)^{-p_{j-1}-p_j-\delta_{ij}} (1+w_j)^{p_{j-1}+p_j+\delta_{ij}-1}. \tag{5.5}$$

Finally, the expression (4.5) for the expansion coefficient becomes

$$[\boldsymbol{\xi}^{\boldsymbol{q}}]\boldsymbol{\eta}^{\boldsymbol{q}'} \equiv [\boldsymbol{x}^{\boldsymbol{m}}\boldsymbol{z}^{\boldsymbol{p}}]\boldsymbol{s}^{\boldsymbol{m}'}\boldsymbol{w}^{\boldsymbol{p}'} = \sum_{i=-1}^{r-1} \mathcal{A}_i^{\boldsymbol{mp}}, \tag{5.6}$$

where

$$\mathcal{A}_i^{\boldsymbol{mp}} \equiv [\boldsymbol{s}^{\boldsymbol{m}}\boldsymbol{w}^{\boldsymbol{p}}]\mathcal{A}_i = \prod_{j=0}^{r-1} \binom{-p_{j-1} - p_j - \delta_{ij}}{m_j - m_j' - \delta_{ij}} \prod_{j=0}^{r-2} \binom{p_{j-1} + p_j + \delta_{ij} - 1}{p_j - p_j' - \sigma_{ij}}. \tag{5.7}$$

6. Discussion

In the present paper we have solved a problem of much physical interest. The spectrum of finite Jacobi matrices appear in many applications: from orthogonal polynomials and nearest-neighbours interaction models to solvable models of quantum mechanics (Lamé polynomials and Bethe ansatz).

What is left for further study are the questions of convergence domains, differential equations and integral representations for the obtained solutions. It would be also interesting to understand the reduction of the jacobian in this and similar cases. The approach used in this work can be generalised to multiparameter spectral problems.

Explicit perturbative solutions have appeared before in the works by Edwin Langmann [12], notably for the multi-dimensional spectral problems related to the Calogero-Sutherland and elliptic Calogero-Moser systems.

References

[1] G.E. Andrews, R. Askey, and R. Roy, *Special functions*, Cambridge Univ. Press, Cambridge (2000). — *Appendix E, pp. 629–631*

[2] A. Panholzer and H. Prodinger, *Kirkman's hypotjesis revisited,* Electron. J. Combin. Number Theory **1**:A5 (2001), 4pp. http://www.integers-ejcnt.org/

[3] R. Birkeland, *Über die Auflösung algebraischer Gleichungen durch hypergeometrische Funktionen,* Math. Zeitschrift **26** (1927) 565–578

[4] K. Mayr, *Über die Auflösung algebraischer Gleichungssysteme durch hypergeometrische Funktionen,* Monatsh. Math. Phy. **45** (1937) 280–313

[5] B. Sturmfels, *Solving algebraic equations in terms of $\mathcal{A}$-hypergeometric series,* Discrete Mathematics **210** (2000) 171–181

[6] I.M. Gel'fand, M.M. Kapranov, and A.V. Zelevinsky, *Hypergeometric functions and toric varieties*, Functional Anal. Appl. **23** (1989) 94–106

[7] I.M. Gel'fand, M.M. Kapranov, and A.V. Zelevinsky, *Generalized Euler integrals and $\mathcal{A}$-hypergeometric functions*, Adv. Math. **84** (1990) 255–271

[8] I.M. Gel'fand, M.M. Kapranov, and A.V. Zelevinsky, *Discriminants, resultants, and multidimensional determinants*, Birkhäuser, Boston, 1994

[9] I.J.Good, *Generalizations to several variables of Lagrange's expansion, with applications to stochastic processes,* Proc. Cambridge Phil. Soc., **56** (1960) 367–380 — *Theorem 10, p. 374*

[10] I.M.Gessel, *A combinatorial proof of the multivariable Lagrange inversion formula,* J. Combin. Theory Ser. A **45** (1987) 178–195

[11] M. Passare and A. Tsikh, *Algebraic equations and hypergeometric series,* In: The legacy of Niels Henrik Abel, 653–672, Springer, Berlin, 2004

[12] E. Langmann, Second quantization of the elliptic Calogero-Sutherland model, Commun. Math. Phys. **247** (2004), 321-351; A perturbative algorithm to solve the (quantum) elliptic Calogero-Sutherland model, arXive: math-ph/0401029

Department of Applied Mathematics, University of Leeds, Leeds LS2 9JT, UK
E-mail address: `V.B.Kuznetsov@leeds.ac.uk`

Department of Mathematics, University of York, York YO10 5DD, UK
E-mail address: `eks2@york.ac.uk`