

**REPORTS
IN
INFORMATICS**

ISSN 0333-3590

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REPORT NO 201

September 2000



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This report has URL <http://www.ii.uib.no/publikasjoner/textrap/pdf/2000-201.pdf>

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This version: 4th September 2000

Abstract

We find a closed-form expression for the Baker–Campbell–Hausdorff formula in the Lie algebra $\mathfrak{so}(3)$, and interpret the formula geometrically in terms of rotation vectors in \mathbb{R}^3 .

Mathematics Subject Classification 2000: 22E60

Key Words: Baker–Campbell–Hausdorff formula, Lie algebra $\mathfrak{so}(3)$, rotations in \mathbb{R}^3

1 Introduction

It is known that the expression for the exponential map taking the Lie algebra $\mathfrak{so}(3)$ into the Lie group $\text{SO}(3)$ has a closed-form expression called Rodrigues' formula [2, p. 291].

$$\exp(\hat{x}) = I + \frac{\sin(\theta)}{\theta} \hat{x} + 2 \frac{\sin^2(\theta/2)}{\theta^2} \hat{x}^2, \quad (1)$$

where $\theta = \|x\|$. In equation (1) the element of $\mathfrak{so}(3)$ is represented as a skew-symmetric matrix \hat{x} , which is obtained from the three-vector x through the hat-map isomorphism:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \longleftrightarrow \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix}.$$

As shown in Appendix B of [1], similar closed-form expressions for other formulae in the case of $\mathfrak{so}(3)$ and $\text{SO}(3)$ are possible to construct.

We will in this note consider the Baker–Campbell–Hausdorff (BCH) formula in the Lie algebra $\mathfrak{so}(3)$, and find an expression in the spirit of (1), albeit, a bit more involved.

2 On the BCH formula in $\mathfrak{so}(3)$

2.1 A closed form expression for the BCH formula in $\mathfrak{so}(3)$

We now recall the definition of the BCH formula. See e.g. [3] for more details. In general, define $\text{BCH}(\hat{u}, \hat{v})$, for $\hat{u}, \hat{v} \in \mathfrak{g}$, where \mathfrak{g} is a Lie algebra, as

$$\exp(\text{BCH}(\hat{u}, \hat{v})) = \exp(\hat{u}) \exp(\hat{v}).$$

For the rest of this note we restrict our attention to the Lie algebra $\mathfrak{so}(3)$, which consists of 3×3 skew-symmetric matrices.

We now state our theorem.

Theorem 2.1 (Expression for BCH in $\mathfrak{so}(3)$)

The BCH formula in $\mathfrak{so}(3)$ has the form

$$\text{BCH}(\hat{u}, \hat{v}) = \alpha \hat{u} + \beta \hat{v} + \gamma [\hat{u}, \hat{v}],$$

where $[\hat{u}, \hat{v}]$ denotes the commutator of \hat{u} and \hat{v} , and α, β , and γ are real constants of the form

$$\alpha = \frac{\sin^{-1}(d)}{d} \frac{a}{\theta}, \quad \beta = \frac{\sin^{-1}(d)}{d} \frac{b}{\phi}, \quad \gamma = \frac{\sin^{-1}(d)}{d} \frac{c}{\theta \phi},$$

where a, b, c , and d are defined as

$$\begin{aligned} a &= \sin(\theta) \cos^2(\phi/2) - \sin(\phi) \sin^2(\theta/2) \cos(\angle(u, v)), \\ b &= \sin(\phi) \cos^2(\theta/2) - \sin(\theta) \sin^2(\phi/2) \cos(\angle(u, v)), \\ c &= \frac{1}{2} \sin(\theta) \sin(\phi) - 2 \sin^2(\theta/2) \sin^2(\phi/2) \cos(\angle(u, v)), \\ d &= \sqrt{a^2 + b^2 + 2ab \cos(\angle(u, v)) + c^2 \sin^2(\angle(u, v))}. \end{aligned}$$

In the above formulae $\theta = \|u\|$, $\phi = \|v\|$, and $\angle(u, v)$ is the angle between the two vectors u and v .

The proof of this theorem is by straight forward calculation. Start off by calculating the product of $\exp(\hat{u}) \exp(\hat{v})$ using Rodrigues' formula (1), resulting in

$$\begin{aligned} A &= \exp(\hat{u}) \exp(\hat{v}) \\ &= I + 2 \frac{\sin^2(\theta/2)}{\theta^2} \hat{u}^2 + 2 \frac{\sin^2(\phi/2)}{\phi^2} \hat{v}^2 \end{aligned} \tag{2}$$

$$+ \frac{\sin(\theta)}{\theta} \hat{u} + \frac{\sin(\phi)}{\phi} \hat{v} + \frac{\sin(\theta) \sin(\phi)}{\theta \phi} \hat{u} \hat{v} + 4 \frac{\sin^2(\theta/2) \sin^2(\phi/2)}{\theta^2 \phi^2} \hat{u}^2 \hat{v}^2 \tag{3}$$

$$+ 2 \frac{\sin(\theta) \sin^2(\phi/2)}{\theta \phi^2} \hat{u} \hat{v}^2 + 2 \frac{\sin(\phi) \sin^2(\theta/2)}{\phi \theta^2} \hat{u}^2 \hat{v}. \tag{4}$$

The terms in A are ordered such that (2) is the symmetric part, and (3-4) is the skew-symmetric part. The reason for this ordering of the terms is that we want to take the logarithm of A in order to determine the expression for $\text{BCH}(\hat{u}, \hat{v})$, and as a result, the symmetric part drops out.

Lemma 2.2 (Logarithm of $A \in \text{SO}(3)$ [1])

The logarithm of $A \in \text{SO}(3)$ is found as follows. Denote by $\hat{z} = (A - A^T)/2$ the skew-symmetric part of A . Then

$$\log(A) = \frac{\sin^{-1}(\|z\|)}{\|z\|} \hat{z}.$$

This lemma is easily proved by staring at the Rodrigues' formula (1), and noticing that I and \hat{x}^2 are symmetric. Note that in MATLAB \sin^{-1} is implemented such that in the case of all positive real parts of the eigenvalues of A , $\sin^{-1} = \text{asin}$, whereas for the case of two equal negative real parts $\sin^{-1} = \pi - \text{asin}$.

We continue the proof of Theorem 2.1 by finding an expression for the skew-symmetric part of A .

$$\begin{aligned} \hat{z} &= (A - A^T)/2 \\ &= \frac{\sin(\theta)}{\theta} \hat{u} + \frac{\sin(\phi)}{\phi} \hat{v} + \frac{1}{2} \frac{\sin(\theta)}{\theta} \frac{\sin(\phi)}{\phi} [\hat{u}, \hat{v}] \\ &\quad + \frac{\sin(\theta)}{\theta} \frac{\sin^2(\phi/2)}{\phi^2} (\hat{u}\hat{v}^2 + \hat{v}^2\hat{u}) + \frac{\sin(\phi)}{\phi} \frac{\sin^2(\theta/2)}{\theta^2} (\hat{u}^2\hat{v} + \hat{v}\hat{u}^2) \\ &\quad + 2 \frac{\sin^2(\theta/2)}{\theta^2} \frac{\sin^2(\phi/2)}{\phi^2} (\hat{u}^2\hat{v}^2 - \hat{v}^2\hat{u}^2). \end{aligned}$$

Now, using the identities

$$\begin{aligned} \hat{u}^2\hat{v} + \hat{v}\hat{u}^2 &= -\theta^2\hat{v} - (u^T v)\hat{u}, \\ \hat{u}^2\hat{v}^2 - \hat{v}^2\hat{u}^2 &= -(u^T v)[\hat{u}, \hat{v}], \\ (u^T v) &= \theta \phi \cos(\angle(u, v)), \end{aligned}$$

the expression for \hat{z} simplifies further.

$$\hat{z} = (\sin(\theta) \cos^2(\phi/2) - \sin(\phi) \sin^2(\theta/2) \cos(\angle(u, v))) \frac{\hat{u}}{\theta} \quad (5)$$

$$+ (\sin(\phi) \cos^2(\theta/2) - \sin(\theta) \sin^2(\phi/2) \cos(\angle(u, v))) \frac{\hat{v}}{\phi} \quad (6)$$

$$+ \left(\frac{1}{2} \sin(\theta) \sin(\phi) - 2 \sin^2(\theta/2) \sin^2(\phi/2) \cos(\angle(u, v)) \right) \left[\frac{\hat{u}}{\theta}, \frac{\hat{v}}{\phi} \right] \quad (7)$$

$$= a\hat{e}_u + b\hat{e}_v + c[\hat{e}_u, \hat{e}_v], \quad (8)$$

where we have defined the unit matrices \hat{e}_x as $\frac{\hat{x}}{\|x\|}$, and the constants a , b , and c by (5), (6), and (7), respectively.

Using (8), it is possible to show that the norm of z is given as

$$\|z\|^2 = a^2 + b^2 + 2ab \cos(\angle(u, v)) + c^2 \sin^2(\angle(u, v)).$$

Scaling \hat{z} with $\sin^{-1}(\|z\|)/\|z\|$, according to Lemma 2.2, and comparing with the constants in Theorem 2.1, we have completed the proof. \square

2.2 Interpreting the BCH formula in $\mathfrak{so}(3)$ as a rotation vector

Interpreting Rodrigues' formula (1) as a rotation around the vector x an angle $\|x\|$, we can interpret $\exp(\hat{u})\exp(\hat{v})$ as the composition of two such rotations. First we rotate around the vector v an angle $\|v\|$, followed by a rotation around the vector u an angle $\|u\|$. Then, $\hat{w} = \text{BCH}(\hat{u}, \hat{v})$ is the vector such that the combined rotation around the vector u and v , is replaced by one rotation around the vector

$$w = \alpha u + \beta v + \gamma u \times v,$$

where the constants are the same as in Theorem 2.1.

3 Conclusion

In this note we have found a closed-form expression for the BCH formula in $\mathfrak{so}(3)$. This adds another member to the family of finite-term expressions for important functions on $\mathfrak{so}(3)$ and $\text{SO}(3)$, such as the exponential map, Cayley map, logarithm, etc.

Acknowledgment

I thank Arne Marthinsen for in part suggesting the present investigation.

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